Homogenization of a reaction-diffusion-advection problem in an evolving micro-domain and including nonlinear boundary conditions

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Abstract

We consider a reaction-diffusion-advection problem in a perforated medium, with nonlinear reactions in the bulk and at the microscopic boundary, and low diffusion scaling. The microstructure changes in time; the microstructural evolution is known *a priori*. The aim of the paper is the rigorous derivation of a homogenized model. We use appropriately scaled function spaces, which allow us to show compactness results, especially regarding the time-derivative and we prove strong two-scale compactness results of Kolmogorov-Simon-type, which allow to pass to the limit in the nonlinear terms. The derived macroscopic model depends on the micro- and the macro-variable, and the evolution of the underlying microstructure is approximated by time- and space-dependent reference elements.

1 Introduction

In this paper, we consider a reaction-diffusion-advection problem in a perforated medium with evolving microstructure (the micro-domain). Such type of problems are encountered as mathematical models for mineral precipitation or biofilm growth in porous media [29, 6, 27, 26], manufacturing of steel [10], or the swelling of mitochondria within biological cells [25]. Typically, in such mathematical models two spatial scales can be encountered: a microscopic scale representing e.g. the scale of pores in a porous medium and at which the processes can be described in detail, and a macroscopic one, representing e.g. the scale of the full domain.

In situations like presented above, the microscopic processes can alter significantly the microstructure, which changes in time. The evolution of the micro-domain then depends on unknown quantities like the concentration of the transported species, or the fluid flow and the pressure. On the other hand, these unknowns do depend on the microstructural evolution, so one speaks about problems involving free boundaries at the microscopic scale. From a numerical

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point of view, such mathematical models are extremely complex, which makes simulations a challenging task.

In this context, a natural approach is to derive upscaled, macroscopic mathematical models approximating the microscopic ones, and describing the averaged behaviour of the quantities. Compared to the models involving a fixed microstructure, the case in which the microscopic sub-domains occupied by the fluid and by the solid phases are separated by a moving interface requires additionally a way to express the microstructural evolution in time. One possibility is to use phase fields as smooth approximations of the characteristic functions of the two above-mentioned sub-domains (see e.g. [24, 7] for a reactive transport model in a porous medium involving precipitation and dissolution). This has the advantage that the moving boundaries are approximated by diffuse transition regions and that the microscopic model is defined in a fixed domain. On the other hand, this introduces curvature effects in the evolution of this diffuse region and, implicitly, of the evolving interface. Furthermore, two different limits have to be considered: one in which the diffuse interface is reduced to a sharp one, and another in which the macroscopic model is obtained from the microscopic one. Choosing the proper balance in this limit process is not trivial, in particular in a mathematically rigorous derivation.

Alternatively, one can consider sharp interfaces separating the phase subdomains. In the one-dimensional case, or for simple geometries like a (thin) strip or a radially symmetric tube, the distance to the lateral boundary can be used to locate the moving interface. For more complex situations like perforated domains, the moving boundaries can be defined as the 0-level set of a function solving a level set equation involving the other model unknowns. In this sense we refer to the micro-model proposed in [29] for crystal precipitation and dissolution in a porous medium. The evolution of microscopic precipitate layer is described through a level-set, and a macroscopic model is derived by a formal asymptotic expansion. In [6], this procedure is extended to account for non-isothermal effects. Similarly, in [27] the level set is used to describe the microscopic growth of a biofilm in a porous medium, and the corresponding macroscopic model is derived by formal homogenization techniques.

For giving a mathematically rigorous derivation of the macroscopic model, and more precise of the convergence of the limiting process, the main difficulties are in proving the existence of a solution for the microscopic, free boundary model, and in obtaining a priori estimates that are sufficient for a two-scale limit. In a first step toward the rigorous homogenization of problems involving free boundaries at the micro scale, here we consider a simplified situation and assume that the microstructural evolution is known *a priori*. In other words, we assume that the movement of the interface can be expressed through a time and space dependent mapping from a reference domain (see also [21, 22, 10]).

In what follows, $\epsilon > 0$ is a small parameter describing the ratio between the typical micro-scale length and the one of the n-dimensional hyper-rectangle Ω. It also represents the periodicity parameter of a fixed, periodically perforated domain Ω_ϵ. At each time t, the micro-domain is $\Omega_\epsilon(t)$ and is obtained as a mapping of Ω_ϵ. More precisely, with $T > 0$ being a finite, maximal time, the mapping $S_\epsilon : [0, T] \times \Omega_\epsilon \to \mathbb{R}^n$ is assumed known and sufficiently regular, and for each t, the microscopic domain (occupied by the fluid) is given as $\Omega_\epsilon(t) = S_\epsilon(t, \Omega_\epsilon)$. In this context, Ω_ϵ has an oscillating boundary Γ_ϵ (see Figure 1). Correspondingly, for each $t \in [0, T]$, the microscopic moving boundary is $\Gamma_\epsilon(t) =$

$S_\epsilon(t, \Gamma_\epsilon)$, and the outer one becomes $\partial\Omega(t)$.

Inside $\Omega_\epsilon(t)$ the microscopic model is a reaction-diffusion-advection equation. The diffusion is assumed of order ϵ^2, and the advection of order ϵ. The boundary $\Gamma_\epsilon(t)$ is moving in time in a manner known *a priori*. This is expressed as a nonlinear Neumann-type boundary condition (see (1) below). The aim of this paper is a mathematically rigorous derivation of a macroscopic model, of which solution approximates the one of the microscopic model. This is done by using rigorous multi-scale techniques such as two-scale convergence [20, 2, 18, 3] and the unfolding method [9, 8], also known as periodic modulation or dilation operator [4, 5, 32]. In doing so, the main challenge is to pass to the limit in the nonlinear terms. Hence, a crucial part of the paper is the derivation of strong two-scale compactness results based only on estimates for the solution of the micro-model.

Since the microstructural evolution is assumed known, to derive the macroscopic model we transform the microscopic problem, defined in the moving domain $\Omega_\epsilon(t)$, to a problem defined on the fixed, reference domain Ω_ϵ. This leads to a change in the coefficients of the equations, which now depend on the transformation S_ϵ between Ω_ϵ and $\Omega_\epsilon(t)$. Especially, the time-derivative is applied to the product $J_\epsilon u_\epsilon$ involving the determinant of the Jacobian of S_ϵ, $J_\epsilon := \det(\nabla S_\epsilon)$. We mention that ϵ-independent estimates are only obtained for the time-derivative $\partial_t(J_\epsilon u_\epsilon)$ and not for $\partial_t u_\epsilon$, and therefore the product $J_\epsilon u_\epsilon$ is used to control the dependence of u_ϵ on the time variable in the homogenization process. Additionally, the regularity of the product $J_\epsilon u_\epsilon$ with respect to time is insufficient to control the nonlinear boundary terms, and therefore refined arguments involving the time-derivative are required for the derivation of strong compactness results. A further challenge in the proof of the strong convergence is due to the small diffusion coefficient (of order ϵ^2), which leads to oscillations of the solution with respect to the spatial variable and thus to a gradient norm of order ϵ^{-1}. Hence, the two-scale limit depends on a macro- and a micro-variable, and standard compactness results used in the case of diffusion coefficients of order 1, like the extension of the solution to the whole domain, see [1], and Aubin-Lions-type compactness arguments, see [16], fail. To overcome these problems, we use the unfolding operator and prove strong convergence of the unfolded sequence in the L^p- sense by using a Banach-valued compactness argument of Kolmogorov-Simon type. These are based on *a priori* estimates for the differences of the shifts with respect to the spatial variable and the control of the time variable via the time-derivative. For the latter, we prove a commuting property for the generalized time-derivative and the unfolding operator using a duality argument, see also [14] where similar techniques were used for reaction-diffusion problems through thin heterogeneous layers.

The derived effective model depends on both micro- and a macro-variables. The macro-variable lies in the fixed domain Ω. In every macro-point of $x \in \Omega$, one has to solve a local cell problem with respect to the micro-variable, and on an evolving reference element. For such cell problems, the macro-variable x enters rather as a parameter. Hence, the evolution of the micro-structure is passed on to the cell problems in the effective model.

Strictly referring to rigorous homogenization results for problems with an evolving microstructure, we mention that even if the microscopic evolution is known *a priori*, there are only a few results available and these are for *linear* problems, at least in the boundary terms. In this sense we mention [21], dealing with a linear reaction-diffusion-advection problem in a two-phase medium, and with diffusion of order ϵ^2 and 1 in the different phases. Similarly, a model related to thermoelasticity in a two-phase domain was considered in [10]. In [22], a reaction-diffusion problem with an evolving volume fraction (in our notation J_ϵ) is treated, where the volume fraction is described by an ordinary differential equation which is not coupled to the solution of the reaction-diffusion equation. In [30, 31] a macroscopic model is derived for a linear double porosity model in a locally periodic medium not evolving in time. A problem on a fixed domain, with diffusion of order ϵ^2 and nonlinear bulk kinetics - however, without the coefficient J_ϵ in the time-derivative - is treated in [17]. The authors determine the strong convergence by estimating the difference between the unfolded equation and the solution to the macroscopic equation, which has to be known *a priori*. However, an additional regularity and compatibility condition for the initial data is required. In [13, 12] a reaction-diffusion system for a two-component connected-disconnected medium was considered for high diffusion (of order 1) and nonlinear interface conditions as in our problem, where in [12] an additional surface equation has been taken into account.

The paper is organized as follows: in Section 2, we introduce the microscopic model in an evolving domain and transform it to a fixed domain. In Section 3, the *a priori* estimates used for the derivation of the macro-model are proved. In Section 4, we establish general strong two-scale compactness results and a convergence result for the time-derivative. The macroscopic model is derived in Section 5. The paper ends with the conclusions in Section 6.

1.1 Original contributions

Here we address the homogenization of a reactive transport model in an evolving perforated domain. The evolution of the microstructure is assumed known a priori. Since nonlinear bulk and surface reactions are considered, the derivation of strong multi-scale compactness results is essential. The main results are:

- The derivation of a priori estimates for the micro-problem with coefficients depending on the transformation between the time-dependent and the fixed domain, and especially of the estimates for the differences between the shifted solution and the solution itself, see Lemma 1 and 2;
- The introduction of the space $\mathcal{H}_\epsilon$ with a weighted Sobolev-norm adapted to the low diffusion scaling (of order ϵ^2), which is the basis for weak and strong two-scale compactness results, see Section 4;
- The commuting property of the generalized time-derivative and the unfolding operator, based on a duality argument, see Proposition 3;
- The general strong two-scale compactness result of Kolmogorov-Simon-type, see Theorem 1;
- The two-scale compactness result for the generalized time-derivative, see Proposition 4;
- The derivation of a homogenized model for a reaction-diffusion-advection equation in an evolving domain, with nonlinear reaction-kinetics and low regularity for the time-derivative, see Section 5.

2 The microscopic model

With $n \in \mathbb{N}$, $n > 1$ and $a, b \in \mathbb{N}^n$ such that $a_i < b_i$ for all $i = 1, \dots n$, we consider the hyper-rectangle $\Omega = (a, b) \subset \mathbb{R}^n$ as the macroscopic domain. Further $\epsilon > 0$ is a small parameters such that $\epsilon^{-1} \in \mathbb{N}$. $\Omega_\epsilon \subset \Omega$ is a fixed, periodically perforated microscopic domain, constructed as follows. With $Y := (0,1)^n$, we let $Y^* \subset Y$ be a connected subdomain with opposite faces matching each other, i. e. for $i = 1, \dots, n$ it holds that

$$\partial Y^* \cap \{x_i = 0\} + e_i = \partial Y^* \cap \{x_i = 1\}.$$

We define $\Gamma := \partial Y^* \setminus \partial Y$ and assume that Γ is a Lipschitz boundary. Further, let $K_\epsilon := \{k \in \mathbb{Z}^n : \epsilon(Y + k) \subset \Omega\}$. Clearly, $\Omega = \operatorname{int}\big(\bigcup_{k \in K_\epsilon} \epsilon\big(\overline{Y} + k\big)\big)$. Now we define Ω_ϵ by

$$\Omega_\epsilon := \operatorname{int}\left(\bigcup_{k \in K_\epsilon} \epsilon\big(\overline{Y^*} + k\big)\right)$$

and the oscillating boundary Γ_ϵ as

$$\Gamma_\epsilon := \partial\Omega_\epsilon \setminus \partial\Omega.$$

By construction, Ω_ϵ is connected and Γ_ϵ is a Lipschitz boundary. We emphasize that the complement $\Omega \setminus \Omega_\epsilon$ may be connected (if $n \geq 3$) or disconnected.

For for $t \in [0, T]$, the evolving domain $\Omega_\epsilon(t)$ and the evolving surface $\Gamma_\epsilon(t)$ are described by a mapping $S_\epsilon : [0, T] \times \Omega \to \mathbb{R}^n$,

$$S_\epsilon(t, \cdot) : \overline{\Omega_\epsilon} \to \overline{\Omega_\epsilon(t)},$$

with $\Gamma_\epsilon(t) = S_\epsilon(t, \Gamma_\epsilon)$, see Figure 1. The Jacobi determinant of S_ϵ is denoted by J_ϵ, i. e. we have $J_\epsilon := \det\big(\nabla S_\epsilon\big)$.

Now we define the non-cylindrical domains Q_ϵ^T and G_ϵ^T by

$$Q_\epsilon^T := \bigcup_{t \in (0,T)} \{t\} \times \Omega_\epsilon(t), \qquad G_\epsilon^T := \bigcup_{t \in (0,T)} \{t\} \times \Gamma_\epsilon(t),$$

and consider **Problem P**, namely to find $\tilde{u}_\epsilon$ satisfying:

$$\begin{aligned}
\partial_t \tilde{u}_\epsilon - \nabla \cdot \big(\epsilon^2 D \nabla \tilde{u}_\epsilon - \epsilon \tilde{q}_\epsilon \tilde{u}_\epsilon\big) &= f(\tilde{u}_\epsilon) && \text{in } Q_\epsilon^T \\
-\big(\epsilon^2 D \nabla \tilde{u}_\epsilon - \epsilon \tilde{q}_\epsilon \tilde{u}_\epsilon\big) \cdot \nu &= -\epsilon g(\tilde{u}_\epsilon) && \text{on } G_\epsilon^T, \\
-\big(\epsilon^2 D \nabla \tilde{u}_\epsilon - \epsilon \tilde{q}_\epsilon \tilde{u}_\epsilon\big) \cdot \nu &= 0 && \text{on } \bigcup_{t \in (0,T)} \{t\} \times \partial\Omega_\epsilon(t) \setminus \Gamma_\epsilon(t), \\
\tilde{u}_\epsilon(0) &= \tilde{u}_\epsilon^0 && \text{in } \Omega_\epsilon(0).
\end{aligned} \tag{1}$$

We use common notations in the functional analysis. For a Banach space X with norm $\|\cdot\|_X$ we also use this notation instead of $\|\cdot\|_{X^n}$ for the norm in the product space $X^n = X \times \ldots \times X$. Also, we use $C > 0$ as a generic constant independent of ϵ. With this, and assuming that all integrals are well defined, the weak form of Problem P (given in (1)) is

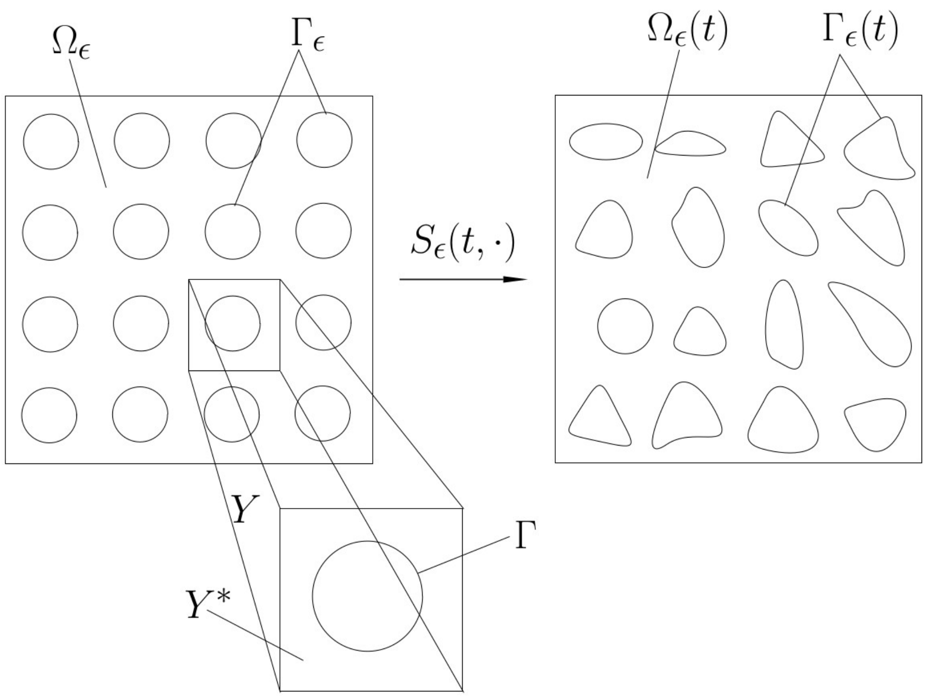

Figure 1: The fixed domain Ω_ϵ and the time-dependent domain $\Omega_\epsilon(t)$ obtained through the mapping $S_\epsilon(t,\cdot)$.

Problem $\mathbf{P}_W$. Find $\tilde{u}_\epsilon \in L^2((0,T), H^1(\Omega_\epsilon(t)))$ such that for all $\phi \in C^1\big(\overline{Q_\epsilon^T}\big)$ with $\phi(T,\cdot) = 0$ it holds that

$$\begin{aligned}
-\int_0^T \int_{\Omega_\epsilon(t)} \tilde{u}_\epsilon \partial_t \phi dx dt &+ \int_0^T \int_{\Omega_\epsilon(t)} \left[\epsilon^2 D \nabla \tilde{u}_\epsilon - \epsilon \tilde{q}_\epsilon \tilde{u}_\epsilon\right] \cdot \nabla \phi dx dt \\
&= \int_0^T \int_{\Omega_\epsilon(t)} f(\tilde{u}_\epsilon)\phi dx dt + \epsilon \int_0^T \int_{\Gamma_\epsilon(t)} g(\tilde{u}_\epsilon)\phi d\sigma dt \\
&\quad + \int_{\Omega_\epsilon(0)} \tilde{u}_\epsilon^0 \phi(0) dx + \int_0^T \int_{\partial_t \Omega_\epsilon(t)} \partial_t S_\epsilon(t, S_\epsilon(t, \cdot_x)^{-1}) \cdot \nu \tilde{u}_\epsilon \phi d\sigma dt.
\end{aligned}$$

Below we state the **assumptions on S_ϵ:**

(A1) $S_\epsilon \in C^1([0,T] \times \overline{\Omega_\epsilon})^n$ with

$$\frac{1}{\epsilon}\|\partial_t S_\epsilon\|_{C^0([0,T]\times\overline{\Omega_\epsilon})} + \|S_\epsilon\|_{C^1([0,T]\times\overline{\Omega_\epsilon})} \leq C.$$

(A2) There exist constants $c_0, C_0 > 0$ independently of ϵ, such that

$$c_0 \leq J_\epsilon \leq C_0.$$

(A3) We have $J_\epsilon \in C^1([0,T] \times \overline{\Omega_\epsilon})$ with

$$\|\partial_t J_\epsilon\|_{L^2((0,T),\mathcal{H}_\epsilon')} + \epsilon\|\nabla J_\epsilon\|_{L^\infty((0,T)\times\Omega_\epsilon)} \leq C.$$

For the definition of the space $\mathcal{H}_\epsilon$ we refer to Section 3.

(A4) For any $\ell \in \mathbb{Z}^n$ it holds that

$$\|J_\epsilon(\cdot_t, \cdot_x + \ell\epsilon) - J_\epsilon\|_{C^0([0,T]\times\overline{\Omega_\epsilon^h})} \leq C|\ell\epsilon|, \text{ and}$$
$$\|\nabla J_\epsilon(\cdot_t, \cdot_x + \ell\epsilon) - \nabla J_\epsilon\|_{L^\infty((0,T)\times\Omega_\epsilon^h)} \leq C|\ell|,$$

with $0 < h \ll 1$ fixed. For the definition of the domain Ω_ϵ^h we refer to Section 3.

(A5) For any $\ell \in \mathbb{Z}^n$ it holds that

$$\|v_\epsilon(\cdot_t, \cdot_x + \ell\epsilon) - v_\epsilon\|_{C^0([0,T]\times\overline{\Omega_\epsilon^h})} \leq C|\ell|\epsilon^2,$$

with $v_\epsilon(t,x) := \nabla S_\epsilon(t,x)^{-1}\partial_t S_\epsilon(t,x)$ and $0 < h \ll 1$ fixed.

(A6) There exists $S_0 \in C^0\big(\overline{\Omega}, C^1\big([0,T]\times\overline{Y^*}\big)\big)^n$ such that $S_0(t,x,\cdot_y)$ is Y-periodic and $S_0(t,x,\cdot) : Y^* \to Y(t,x) := \mathrm{R}(\mathrm{S_0(t,x,\cdot)})$ (the range of $S_0(t,x,\cdot_y)$) is a C^1-diffeomorphism and (for the definition of the two-scale convergence see Section 4)

$$\begin{aligned}
S_\epsilon(t,x) &\to x && \text{strongly in the two-scale sense,}\\
\nabla S_\epsilon(t,x) &\to \nabla_y S_0(t,x,y) && \text{strongly in the two-scale sense,}\\
\nabla S_\epsilon^{-1}(t,x) &\to \nabla_y S_0^{-1}(t,x,y) && \text{strongly in the two-scale sense,}\\
\epsilon^{-1}\partial_t S_\epsilon(t,x,y) &\to \partial_t S_0(t,x,y) && \text{strongly in the two-scale sense.}
\end{aligned}$$

Especially, it holds that $J_\epsilon \to J_0 := \det \nabla_y S_0$ strongly in the two-scale sense. We emphasize that due to the estimates in Assumption (A1) the strong two-scale convergences in (A6) hold with respect to every L^p-norm for $p \in [1,\infty)$ arbitrary large. For the derivation of the macroscopic model we need the strong convergence of $\mathcal{T}_\epsilon J_\epsilon$, where $\mathcal{T}_\epsilon$ denotes the unfolding operator, see Section 4.

In the following we give an example of a transformation S_ϵ, for which the assumptions (A1) - (A6) are fulfilled.

Example 1. *Let us consider that $Y \setminus \overline{Y^*}$ is strictly included in Y, i. e. Γ is a compact subset of Y. Furthermore, let $\omega(t,x,y)$ with $\omega : [0,T]\times\overline{\Omega}\times\overline{Y} \to \mathbb{R}$ be a smooth function, which is Y-periodic with respect to variable y. We assume that for every $(t,x) \in [0,T]\times\overline{\Omega}$ the set*

$$\Gamma(t,x) := \{y + \omega(t,x,y)\nu_0(y) \,:\, y \in \Gamma\} \subset Y \tag{2}$$

is a closed C^2-manifold with $\Gamma(0,x) = \Gamma$. Here ν_0 denotes the outer unit normal vector with respect to Y^*. The properties of $\Gamma(t,x)$ imply the existence of cubes W_o and W_i, such that $W_i \subset W_o \subset Y$ and

$$\Lambda := \bigcup_{(t,x)\in[0,T]\times\overline{\Omega}} \Gamma(t,x) \subset W_o \setminus \overline{W_i}.$$

Let us define a cut-off function $\chi_0 \in C_0^\infty\big(W_o \setminus \overline{W_i}\big)$ with $0 \leq \chi_0 \leq 1$ and $\chi_0 = 1$ in Λ. Now, we define the transformation S_ϵ by

$$S_\epsilon(t,x) := x + \epsilon\omega\left(t, \left[\frac{x}{\epsilon}\right], \frac{x}{\epsilon}\right)\chi_0\left(\frac{x}{\epsilon}\right)\nu_0\left(\frac{x}{\epsilon}\right).$$

This means, that the moving interface $\Gamma_\epsilon(t)$ can be described locally on every microscopic cell. With S_0 defined by

$$S_0(t, x, y) := y + \omega(t, x, y)\chi_0(y)\nu_0(y),$$

the sequence S_ϵ fulfills the properties (A1) - (A6).

Remark 1.

(i) In applications like the ones mentioned in the introduction, the transformation S_ϵ is not known a priori, but is itself a model unknown. For example, when considering dissolution and precipitation in a porous medium, the evolution of $\Gamma_\epsilon(t)$ is determined by the two processes named above, which depend on the concentration of the solute at the pore walls. Conversely, $\Omega_\epsilon(t)$ is the domain in which the solute transport model component is defined, so the solute concentration depends on the evolution of the free boundary $\Gamma_\epsilon(t)$. This leads to a microscopic, moving interface problem. From the evolution of $\Gamma_\epsilon(t)$ the transformation S_ϵ can be constructed for example via the Hanzawa-transformation [15], see also the monograph [23] for an overview on this topic. Regarding Example 1, the crucial point is to find the function ω and to guarantee the condition (2) *together with the regularity of $\Gamma(t, x)$. Often this is only possible locally with respect to time, which rises additional difficulties in the homogenization procedure.*

(ii) We emphasize that the Assumption (A1) on the transformation S_ϵ implies that the microscopic deformations taken into account are small at the micro-scale and, consequently, at the macro-scale as well. In other words, the displacement of every point with respect to its initial configuration is of order ϵ.

Finally, we state the **assumptions on the data:**

(AD1) It holds that $\tilde{q}_\epsilon \in L^\infty(Q_\epsilon^T)^n$ and

$$\|\tilde{q}_\epsilon\|_{L^\infty(Q_\epsilon^T)} \leq C.$$

Further, we assume that for any $\ell \in \mathbb{Z}^n$ it holds that

$$\|\tilde{q}_\epsilon \circ S_\epsilon(\cdot_t, \cdot_x + \ell\epsilon) - \tilde{q}_\epsilon \circ S_\epsilon\|_{L^\infty((0,T)\times\Omega_\epsilon^h)} \leq C|\ell\epsilon|.$$

There exists a Y-periodic function $q_0 \in L^\infty((0, T) \times \Omega \times Y^*)$ such that

$$\tilde{q}_\epsilon \circ S_\epsilon \to q_0 \quad \text{in the two-scale sense.}$$

(AD2) It holds that f, $g \in L^2((0, T) \times \mathbb{R})$, and for almost every $t \in (0, T)$ the functions $z \mapsto f(t, z)$ and $z \mapsto g(t, z)$ are globally Lipschitz-continuous uniformly with respect to t.

(AD3) The diffusion coefficient $D \in \mathbb{R}^{n\times n}$ is symmetric and positive.

(AD4) The initial values $u_\epsilon^0 \in L^2(\Omega_\epsilon)$ are bounded uniformly with respect to ϵ and for any $\ell \in \mathbb{Z}^n$ such that $|\ell\epsilon| < h$ with $0 < h \ll 1$ fixed, it holds that

$$\|u_\epsilon^0(\cdot + \ell\epsilon) - u_\epsilon^0\|_{L^2(\Omega_\epsilon^h)} \overset{\ell\epsilon\to 0}{\longrightarrow} 0.$$

Further, there exists $u^0 \in L^2(\Omega \times Y^*)$, such that $u_\epsilon^0 \to u^0$ in the two-scale sense.

Using the mapping S_ϵ, we transform the problem in (1) to the fixed domain Ω_ϵ. Let us define

$$u_\epsilon : (0,T)\times\Omega_\epsilon \to \mathbb{R}, \quad u_\epsilon(t,x) := \tilde{u}_\epsilon(t, S_\epsilon(t,x))$$

and for $(t,x)\in(0,T)\times\Omega_\epsilon$

$$\begin{aligned} D_\epsilon(t,x) &:= \nabla S_\epsilon(t,x)^{-1} D \nabla S_\epsilon(t,x)^{-T}, \\ q_\epsilon(t,x) &:= \nabla S_\epsilon(t,x)^{-1}\tilde{q}_\epsilon(S_\epsilon(t,x)), \\ v_\epsilon(t,x) &:= \nabla S_\epsilon(t,x)^{-1}\partial_t S_\epsilon(t,x). \end{aligned}$$

Then, on the fixed domain Ω_ϵ we define the transformed **Problem $\mathbf{P}_T$**, namely to find u_ϵ satisfying

$$\begin{aligned} \partial_t\big(J_\epsilon u_\epsilon\big) - \nabla\cdot\big(\epsilon^2 J_\epsilon D_\epsilon \nabla u_\epsilon - \epsilon J_\epsilon q_\epsilon u_\epsilon + J_\epsilon v_\epsilon u_\epsilon\big) &= J_\epsilon f(u_\epsilon) && \text{in } (0,T)\times\Omega_\epsilon, \\ -\big(\epsilon^2 J_\epsilon D_\epsilon \nabla u_\epsilon - \epsilon J_\epsilon q_\epsilon u_\epsilon\big)\cdot\nu &= -\epsilon J_\epsilon g(u_\epsilon) && \text{on } (0,T)\times\Gamma_\epsilon, \\ -\big(\epsilon^2 J_\epsilon D_\epsilon \nabla u_\epsilon - \epsilon J_\epsilon q_\epsilon u_\epsilon\big)\cdot\nu &= 0 && \text{on } (0,T)\times\partial\Omega, \\ u_\epsilon(0) &= u_\epsilon^0 && \text{in } \Omega_\epsilon, \end{aligned} \tag{3}$$

with $u_\epsilon^0(x) := \tilde{u}_\epsilon^0(S_\epsilon(0,x))$.

In the following, given a bounded domain $U\subset\mathbb{R}^n$, the duality pairing between $H^1(U)'$ and $H^1(U)$ is denoted by $\langle\cdot,\cdot\rangle_U$. With this we define a weak solution of Problem P_T (introduced in (3)):

Definition 1. *A weak solution of the micro-problem P_T is a function $u_\epsilon \in L^2((0,T),H^1(\Omega_\epsilon))$, such that $\partial_t(J_\epsilon u_\epsilon)\in L^2((0,T),H^1(\Omega_\epsilon)'))$, $u_\epsilon(0)=u_\epsilon^0$, and for every $\phi\in H^1(\Omega_\epsilon)$ and almost every $t\in(0,T)$ it holds that*

$$\begin{aligned} \big\langle \partial_t(J_\epsilon u_\epsilon),\phi\big\rangle_{\Omega_\epsilon} &+ \int_{\Omega_\epsilon} \big[\epsilon^2 J_\epsilon D_\epsilon\nabla u_\epsilon - \epsilon J_\epsilon q_\epsilon u_\epsilon + J_\epsilon v_\epsilon u_\epsilon\big)\cdot\nabla\phi dx \\ &= \int_{\Omega_\epsilon} J_\epsilon f(u_\epsilon)\phi dx + \epsilon\int_{\Gamma_\epsilon} J_\epsilon g(u_\epsilon)\phi d\sigma + \int_{\partial\Omega_\epsilon} J_\epsilon v_\epsilon\cdot\nu u_\epsilon\phi d\sigma. \end{aligned} \tag{4}$$

The regularity of $\partial_t(J_\epsilon u_\epsilon)$ and the regularity of J_ϵ immediately imply $\partial_t u_\epsilon \in L^2((0,T),H^1(\Omega_\epsilon)')$. However, as will be seen below the estimates for the norm of $\partial_t u_\epsilon$ are not uniform with respect to ϵ, therefore we work with the product $J_\epsilon u_\epsilon$. We start with

Proposition 1. *There exists a unique weak solution of Problem* (3).

Proof. This follows e.g. by applying the Galerkin-method, and after obtaining estimates that are similar to the ones in Lemma 1. We omit the details. □

3 *A priori* estimates

We first introduce an $H^1(\Omega_\epsilon)$-equivalent norm, which is more adapted to the ϵ-scaling in the problem. We denote by $\mathcal{H}_\epsilon$ the space of $H^1(\Omega_\epsilon)$ functions, equipped with the inner product

$$(u_\epsilon, v_\epsilon)_{\mathcal{H}_\epsilon} := (u_\epsilon, v_\epsilon)_{L^2(\Omega_\epsilon)} + \epsilon^2(\nabla u_\epsilon, \nabla v_\epsilon)_{L^2(\Omega_\epsilon)}. \tag{5}$$

The associated norm is denoted by $\|\cdot\|_{\mathcal{H}_\epsilon}$. The spaces $H^1(\Omega_\epsilon)$ and $\mathcal{H}_\epsilon$ are isomorphic with equivalent norms. More precisely,

$$\|\phi_\epsilon\|_{\mathcal{H}_\epsilon} \le \|\phi_\epsilon\|_{H^1(\Omega_\epsilon)} \le \epsilon^{-1}\|\phi_\epsilon\|_{\mathcal{H}_\epsilon} \quad \text{for all } \phi_\epsilon \in H^1(\Omega_\epsilon).$$

We have the embedding

$$L^2((0,T),\mathcal{H}_\epsilon') \hookrightarrow L^2((0,T),H^1(\Omega_\epsilon)'),$$

and for $F_\epsilon \in L^2((0,T),\mathcal{H}_\epsilon')$ it holds that

$$\|F_\epsilon\|_{L^2((0,T),H^1(\Omega_\epsilon)')} \le \|F_\epsilon\|_{L^2((0,T),\mathcal{H}_\epsilon')}.$$

Especially, for any $w_\epsilon \in L^2((0,T),\mathcal{H}_\epsilon) \cap H^1((0,T),\mathcal{H}_\epsilon')$ it holds that

$$\langle \partial_t w_\epsilon, \phi_\epsilon\rangle_{\mathcal{H}_\epsilon',\mathcal{H}_\epsilon} = \langle \partial_t w_\epsilon, \phi_\epsilon\rangle_{H^1(\Omega_\epsilon)',H^1(\Omega_\epsilon)} \quad \text{for all } \phi_\epsilon \in H^1(\Omega_\epsilon).$$

Clearly, the converse embedding also holds: if $F_\epsilon \in L^2((0,T),H^1(\Omega_\epsilon)')$ then $F_\epsilon \in L^2((0,T),\mathcal{H}_\epsilon')$ as well. However, in this case the estimates of the operator norms in the embedding depend badly on ϵ,

$$\|F_\epsilon\|_{L^2((0,T),\mathcal{H}_\epsilon')} \le \epsilon^{-1}\|F_\epsilon\|_{L^2((0,T),H^1(\Omega_\epsilon)')}.$$

We will see in Section 4 that the norm on $\mathcal{H}_\epsilon'$ is the appropriate norm for the time-derivative to obtain two-scale compactness results.

The following Lemma provides some basic a priori estimates for the solution u_ϵ and the product $J_\epsilon u_\epsilon$.

Lemma 1. *A constant $C > 0$ not depending on ϵ exists such that for the weak solution u_ϵ of Problem P_T one has*

$$\|u_\epsilon\|_{L^\infty((0,T),L^2(\Omega_\epsilon))} + \|u_\epsilon\|_{L^2((0,T),\mathcal{H}_\epsilon)} \le C, \tag{6a}$$

$$\|\partial_t(J_\epsilon u_\epsilon)\|_{L^2((0,T),\mathcal{H}_\epsilon')} + \|J_\epsilon u_\epsilon\|_{L^2((0,T),\mathcal{H}_\epsilon)} \le C. \tag{6b}$$

Proof. We choose u_ϵ as a test function in equation (4). The positivity of D and the properties of S_ϵ imply the coercivity of D_ϵ. Using $c_0 \le J_\epsilon$ and the formulas (remember that $\partial_t J_\epsilon$ is a C^0-function)

$$\big\langle \partial_t(J_\epsilon u_\epsilon), u_\epsilon\big\rangle_{\Omega_\epsilon} = \frac{1}{2}\frac{d}{dt}\|\sqrt{J_\epsilon}u_\epsilon\|^2_{L^2(\Omega_\epsilon)} + \frac{1}{2}\int_{\Omega_\epsilon} \partial_t J_\epsilon u_\epsilon^2 dx,$$

and (since $\partial_t J_\epsilon = \nabla\cdot(J_\epsilon v_\epsilon)$)

$$\int_{\Omega_\epsilon} J_\epsilon u_\epsilon v_\epsilon \cdot \nabla u_\epsilon dx = -\frac{1}{2}\int_{\Omega_\epsilon} \partial_t J_\epsilon u_\epsilon^2 dx + \frac{1}{2}\int_{\partial\Omega_\epsilon} J_\epsilon v_\epsilon \cdot \nu u_\epsilon^2 d\sigma,$$

we obtain for a constant $d > 0$

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\sqrt{J_\epsilon}u_\epsilon\|^2_{L^2(\Omega_\epsilon)} + d\epsilon^2\|\nabla u_\epsilon\|^2_{L^2(\Omega_\epsilon)} \\
&\le -\frac{1}{2}\int_{\Omega_\epsilon} \partial_t J_\epsilon u_\epsilon^2 dx + \epsilon\int_{\Omega_\epsilon} J_\epsilon u_\epsilon q_\epsilon \cdot \nabla u_\epsilon dx - \int_{\Omega_\epsilon} J_\epsilon u_\epsilon v_\epsilon \cdot \nabla u_\epsilon dx \\
&\quad + \int_{\Omega_\epsilon} J_\epsilon f(u_\epsilon) u_\epsilon dx + \epsilon\int_{\Gamma_\epsilon} J_\epsilon h(u_\epsilon) u_\epsilon d\sigma + \int_{\partial\Omega_\epsilon} J_\epsilon v_\epsilon \cdot \nu u_\epsilon^2 d\sigma \\
&= \epsilon\int_{\Omega_\epsilon} J_\epsilon u_\epsilon q_\epsilon \cdot \nabla u_\epsilon dx + \int_{\Omega_\epsilon} J_\epsilon f(u_\epsilon) u_\epsilon dx \\
&\quad + \epsilon\int_{\Gamma_\epsilon} J_\epsilon g(u_\epsilon) u_\epsilon d\sigma + \frac{1}{2}\int_{\partial\Omega_\epsilon} J_\epsilon v_\epsilon \cdot \nu u_\epsilon^2 d\sigma =: \sum_{j=1}^{4} A_\epsilon^{(j)}.
\end{aligned}$$

We continue with the terms on the right-hand side. For any $\theta > 0$, a $C(\theta) > 0$ exists such that for the first term one has

$$A_\epsilon^{(1)} \leq C\epsilon \|J_\epsilon q_\epsilon\|_{L^\infty(\Omega_\epsilon)} \|u_\epsilon\|_{L^2(\Omega_\epsilon)} \|\nabla u_\epsilon\|_{L^2(\Omega_\epsilon)} \leq C(\theta)\|u_\epsilon\|^2_{L^2(\Omega_\epsilon)} + \theta\epsilon^2 \|\nabla u_\epsilon\|^2_{L^2(\Omega_\epsilon)}.$$

Concerning the nonlinear terms, we only consider the boundary integral. All others can be treated in a similar way. We use the trace inequality for periodically perforated domains, stating that for every $\theta > 0$, there exists $C(\theta) > 0$, such that for all $w_\epsilon \in H^1(\Omega_\epsilon)$ one has

$$\epsilon \|w_\epsilon\|^2_{L^2(\partial\Omega_\epsilon)} \leq C(\theta)\|w_\epsilon\|^2_{L^2(\Omega_\epsilon)} + \theta\epsilon^2 \|\nabla u_\epsilon\|^2_{L^2(\Omega_\epsilon)}. \tag{7}$$

Together with the Lipschitz continuity of g, this implies that

$$\begin{aligned} A_\epsilon^{(3)} &\leq C\epsilon \int_{\Gamma_\epsilon} 1 + |u_\epsilon|^2 d\sigma \leq C\big(1 + \epsilon\|u_\epsilon\|^2_{L^2(\Gamma_\epsilon)}\big) \\ &\leq C(\theta)\big(1 + \|u_\epsilon\|^2_{L^2(\Omega_\epsilon)}\big) + \theta\epsilon^2\|\nabla u_\epsilon\|^2_{L^2(\Omega_\epsilon)}. \end{aligned}$$

For $A_\epsilon^{(4)}$, we use similar arguments as above and Assumption (A1) to obtain

$$A_\epsilon^{(4)} \leq C\epsilon\|u_\epsilon\|^2_{L^2(\partial\Omega_\epsilon)} \leq C(\theta)\|u_\epsilon\|^2_{L^2(\Omega_\epsilon)} + \theta\epsilon^2\|\nabla u_\epsilon\|^2_{L^2(\Omega_\epsilon)}.$$

Choosing θ small enough, the terms on the right including the gradients can be absorbed by similar terms on the left. Using the Gronwall-inequality and the $\mathcal{H}_\epsilon$-norm induced by the inner product (5), we obtain inequality (6a).

The $L^2((0,T),\mathcal{H}_\epsilon)$-estimate for $J_\epsilon u_\epsilon$ follows directly from (6a) and the properties of J_ϵ. To prove the inequality for the time-derivative $\partial_t(J_\epsilon u_\epsilon)$, we choose $\phi \in \mathcal{H}_\epsilon$ with $\|\phi\|_{\mathcal{H}_\epsilon} \leq 1$ as a test function in (4), and apply similar arguments as above. □

We emphasize that in the proof above we used the C^1-regularity of J_ϵ, but we have not assumed a uniform bound with respect to ϵ for the C^1-norm. We overcome this problem by using the equality $\partial_t J_\epsilon = \nabla \cdot (J_\epsilon v_\epsilon)$.

Remark 2. *Due to the C^1-regularity of J_ϵ and the product rule we have*

$$\partial_t u_\epsilon = J_\epsilon^{-1}\partial_t(J_\epsilon u_\epsilon) - J_\epsilon^{-1}u_\epsilon\partial_t J_\epsilon \in L^2((0,T),\mathcal{H}_\epsilon') \hookrightarrow L^2((0,T),H^1(\Omega_\epsilon)').$$

However, we only have a uniform bound for the $L^2((0,T),\mathcal{H}_\epsilon')$-norm of $\partial_t J_\epsilon$, see (A3). Therefore, we obtain no bounds for $\partial_t u_\epsilon$ in the $L^2((0,T),\mathcal{H}_\epsilon')$ norm (or in $L^2((0,T),H^1(\Omega_\epsilon)')$) that are uniform in ϵ. Under the additional assumption

$$\|\partial_t J_\epsilon\|_{L^\infty((0,T)\times\Omega_\epsilon)} \leq C,$$

from the results above one also gets

$$\|\partial_t u_\epsilon\|_{L^2((0,T),\mathcal{H}_\epsilon')} \leq C.$$

3.1 Estimates for the shifted functions

To prove strong compactness results which are needed to pass to the limit in the nonlinear terms, we need additional a priori estimates for the difference between shifted functions and the function itself. For $h > 0$ let us define the set $\Omega^h := \{x \in \Omega \, : \, \mathrm{dist}(x, \partial\Omega) > h\}$. Further we define

$$K_\epsilon^h := \{k \in \mathbb{Z}^n \, : \, \epsilon(Y + k) \subset \Omega^h\},$$

$$\Omega_\epsilon^h := \mathrm{int}\left(\bigcup_{k \in K_\epsilon^h} \epsilon\big(\overline{Y^*} + k\big)\right),$$

$$\Gamma_\epsilon^h := \mathrm{int}\left(\bigcup_{k \in K_\epsilon^h} \epsilon\big(\overline{\Gamma} + k\big)\right).$$

Then, for $\ell \in \mathbb{Z}^n$ with $|\ell\epsilon| < h$ we define for an arbitrary function $v_\epsilon : \Omega_\epsilon \to \mathbb{R}$ the shifted function

$$v_\epsilon^\ell(x) := v_\epsilon(x + \ell\epsilon),$$

and the difference between the shifted function and the function

$$\delta v_\epsilon(x) := v_\epsilon^\ell(x) - v_\epsilon(x) = v_\epsilon(x + \ell\epsilon) - v_\epsilon(x).$$

In the following, we want to estimate $\delta(J_\epsilon u_\epsilon)$. Therefore, we need the variational equation for the shifted function u_ϵ^ℓ, respectively $J_\epsilon^\ell u_\epsilon^\ell$. Due to the low regularity for the time-derivative $\partial_t(J_\epsilon u_\epsilon) \in L^2((0,T), H^1(\Omega_\epsilon)')$, it is not obvious how to obtain the time-derivative of $(J_\epsilon u_\epsilon)^\ell$, since this function is only defined on Ω_ϵ^h. Therefore, we argue in the following way: First of all, we define the space

$$\mathcal{H}_0^1(\Omega_\epsilon^h) := \left\{\phi_\epsilon \in H^1(\Omega_\epsilon^h) \, : \, \phi_\epsilon = 0 \text{ on } \partial\Omega_\epsilon^h \setminus \Gamma_\epsilon^h\right\}$$

of Sobolev functions with zero trace on the outer boundary of Ω_ϵ^h. For $v_\epsilon \in H^1((0,T), H^1(\Omega_\epsilon)')$ it holds that the restriction to Ω_ϵ^h fulfills $v_\epsilon \in H^1((0,T), \mathcal{H}_0^1(\Omega_\epsilon^h)')$ with

$$\langle \partial_t v_\epsilon, \phi \rangle_{\mathcal{H}_0^1(\Omega_\epsilon^h)', \mathcal{H}_0^1(\Omega_\epsilon^h)} = \langle \partial_t v_\epsilon, \widetilde{\phi} \rangle_{H^1(\Omega_\epsilon)', H^1(\Omega_\epsilon)},$$

for all $\phi \in \mathcal{H}_0^1(\Omega_\epsilon^h)$ and $\widetilde{\phi}$ denotes the zero extension of ϕ to Ω_ϵ. Further, we have

$$\langle \partial_t v_\epsilon^\ell, \phi \rangle_{\mathcal{H}_0^1(\Omega_\epsilon^h)', \mathcal{H}_0^1(\Omega_\epsilon^h)} = \langle \partial_t v_\epsilon, \widetilde{\phi}^{-l} \rangle_{H^1(\Omega_\epsilon)', H^1(\Omega_\epsilon)}.$$

Finally, the space $\mathcal{H}_\epsilon^h$ is defined in the same way as $\mathcal{H}_\epsilon$, with Ω_ϵ^h replacing Ω_ϵ.

Lemma 2. *Let u_ϵ be the sequence of weak solutions of Problem P_T (stated in (3)), $0 < h \ll 1$ and $\ell \in \mathbb{Z}^n$ such that $|\ell\epsilon| < h$. Then it holds*

$$\|\delta(J_\epsilon u_\epsilon)\|_{L^2((0,T),\mathcal{H}_\epsilon^{2h})} \leq C\|\delta(J_\epsilon(0)u_\epsilon^0)\|_{L^2(\Omega_\epsilon^h)} + C|\ell\epsilon| + C\sqrt{\epsilon}.$$

Proof. For the ease of writing we use the notation

$$w_\epsilon(t,x) := J_\epsilon(t,x)u_\epsilon(t,x).$$

Using the product rule gives

$$J_\epsilon D_\epsilon \nabla u_\epsilon = D_\epsilon \nabla w_\epsilon - w_\epsilon \frac{D_\epsilon}{J_\epsilon} \nabla J_\epsilon.$$

An elemental calculation gives us the following variational equation for all $\phi \in \mathcal{H}_0^1(\Omega_\epsilon^h)$ and almost everywhere in $(0,T)$

$$
\begin{aligned}
&\big\langle \partial_t \delta w_\epsilon, \phi \big\rangle_{\mathcal{H}_0^1(\Omega_\epsilon^h)', \mathcal{H}_0^1(\Omega_\epsilon^h)} + \epsilon^2 \int_{\Omega_\epsilon^h} D_\epsilon \nabla \delta w_\epsilon \cdot \nabla \phi dx \\
=& \langle \partial_t w_\epsilon^\ell, \phi \rangle_{\mathcal{H}_0^1(\Omega_\epsilon^h)', \mathcal{H}_0^1(\Omega_\epsilon^h)} + \epsilon^2 \int_{\Omega_\epsilon^h} D_\epsilon^\ell \nabla w_\epsilon^\ell \cdot \nabla \phi dx - \epsilon^2 \int_{\Omega_\epsilon^h} \delta D_\epsilon \nabla w_\epsilon^\ell \cdot \nabla \phi dx \\
&- \langle \partial_t w_\epsilon, \phi \rangle_{\mathcal{H}_0^1(\Omega_\epsilon^h)', \mathcal{H}_0^1(\Omega_\epsilon^h)} - \epsilon^2 \int_{\Omega_\epsilon^h} D_\epsilon \nabla w_\epsilon \cdot \nabla \phi dx \\
=& \langle \partial_t w_\epsilon^\ell, \phi \rangle_{\mathcal{H}_0^1(\Omega_\epsilon^h)', \mathcal{H}_0^1(\Omega_\epsilon^h)} + \epsilon^2 \int_{\Omega_\epsilon^h} J_\epsilon^\ell D_\epsilon^\ell \nabla u_\epsilon^\ell \cdot \phi dx \\
&- \langle \partial w_\epsilon, \phi \rangle_{\mathcal{H}_0^1(\Omega_\epsilon^h)', \mathcal{H}_0^1(\Omega_\epsilon^h)} - \epsilon^2 \int_{\Omega_\epsilon^h} J_\epsilon D_\epsilon \nabla u_\epsilon \cdot \phi dx \\
&- \epsilon^2 \int_{\Omega_\epsilon^h} \delta D_\epsilon \nabla w_\epsilon^\ell \cdot \phi dx + \epsilon^2 \int_{\Omega_\epsilon^h} \delta \left(w_\epsilon \frac{D_\epsilon}{J_\epsilon} \nabla J_\epsilon \right) \cdot \nabla \phi dx \\
=& - \epsilon^2 \int_{\Omega_\epsilon^h} \delta D_\epsilon \nabla w_\epsilon^\ell \cdot \nabla \phi dx + \epsilon^2 \int_{\Omega_\epsilon^h} \delta w_\epsilon \frac{D_\epsilon}{J_\epsilon} \nabla J_\epsilon \cdot \nabla \phi dx \\
&+ \epsilon^2 \int_{\Omega_\epsilon^h} \delta \left(\frac{D_\epsilon}{J_\epsilon} \nabla J_\epsilon \right) w_\epsilon^\ell \cdot \nabla \phi dx + \epsilon \int_{\Omega_\epsilon^h} w_\epsilon^\ell \delta q_\epsilon \cdot \nabla \phi dx \\
&+ \epsilon \int_{\Omega_\epsilon^h} \delta w_\epsilon q_\epsilon \cdot \nabla \phi dx - \int_{\Omega_\epsilon^h} w_\epsilon^\ell \delta v_\epsilon \cdot \nabla \phi dx \\
&- \int_{\Omega_\epsilon^h} \delta w_\epsilon v_\epsilon \cdot \nabla \phi dx + \int_{\Omega_\epsilon^h} \big(J_\epsilon^\ell f(u_\epsilon^\ell) - J_\epsilon f(u_\epsilon) \big) \phi dx \\
&+ \epsilon \int_{\Gamma_\epsilon^h} \big(J_\epsilon^\ell g(u_\epsilon^\ell) - J_\epsilon g(u_\epsilon) \big) \phi d\sigma \\
&+ \int_{\Gamma_\epsilon^h} \big(J_\epsilon^\ell v_\epsilon^\ell \cdot \nu u_\epsilon^\ell - J_\epsilon v_\epsilon \cdot \nu u_\epsilon \big) \phi d\sigma =: \sum_{j=1}^{10} A_\epsilon^{(j)}.
\end{aligned}
\tag{8}
$$

Now, we choose $\phi = \eta^2 \delta w_\epsilon$, where $\eta \in C^\infty\big(\overline{\Omega_\epsilon^h}\big)$ is a cut-off function with $0 \leq \eta \leq 1$, $\eta = 1$ in $\overline{\Omega_\epsilon^{2h}}$ and $\eta = 0$ in $\Omega_\epsilon \setminus \Omega_\epsilon^h$. This implies

$$
\big\langle \partial_t \delta w_\epsilon, \eta^2 \delta w_\epsilon \big\rangle_{\mathcal{H}_0^1(\Omega_\epsilon^h)', \mathcal{H}_0^1(\Omega_\epsilon^h)} = \frac{1}{2} \frac{d}{dt} \|\eta \delta w_\epsilon\|_{L^2(\Omega_\epsilon)}^2
$$

and the coercivity of D_ϵ implies the existence of a constant $d > 0$ such that (we extend every function by zero outside Ω_ϵ)

$$
\begin{aligned}
\epsilon^2 &\int_{\Omega_\epsilon^h} D_\epsilon \nabla \delta w_\epsilon \cdot \nabla \big(\eta^2 \delta w_\epsilon \big) dx \\
&= \epsilon^2 \int_{\Omega_\epsilon} D_\epsilon \eta^2 \nabla \delta w_\epsilon \cdot \nabla \delta w_\epsilon dx + 2\epsilon^2 \int_{\Omega_\epsilon} \eta \delta w_\epsilon D_\epsilon \nabla \delta w_\epsilon \cdot \nabla \eta dx \\
&\geq \epsilon^2 d \|\eta \nabla \delta w_\epsilon\|_{L^2(\Omega_\epsilon)}^2 + 2\epsilon^2 \int_{\Omega_\epsilon} \eta \delta w_\epsilon D_\epsilon \nabla \delta w_\epsilon \cdot \nabla \eta dx.
\end{aligned}
$$

For the last term we use Lemma 1 to obtain for every $\theta > 0$ a constant $C(\theta) > 0$

such that

$$\epsilon^2 \left| \int_{\Omega_\epsilon} \eta \delta w_\epsilon D_\epsilon \nabla \delta w_\epsilon \cdot \nabla \eta dx \right| \le C(\theta)\epsilon^2 \|\delta w_\epsilon\|^2_{L^2(\Omega_\epsilon^h)} + \theta \epsilon^2 \|\eta \nabla \delta w_\epsilon\|^2_{L^2(\Omega_\epsilon)}$$
$$\le C(\theta)\epsilon^2 + \theta\epsilon^2 \|\eta\nabla\delta w_\epsilon\|^2_{L^2(\Omega_\epsilon)}.$$

Now we estimate the terms $A_\epsilon^{(j)}$. From the assumptions on S_ϵ we obtain $\|\delta D_\epsilon\|_{L^\infty((0,T)\times\Omega_\epsilon^h)} \le C|\ell\epsilon|$. This, together with the a priori estimates in Lemma 1 and after integration with respect to time, implies that for arbitrary $\theta > 0$, a constant $C(\theta) > 0$ exists s.t.

$$\begin{aligned}
\int_0^t A_\epsilon^{(1)} ds = &-\epsilon^2 \int_0^t \int_{\Omega_\epsilon^h} \eta^2 \delta D_\epsilon \nabla w_\epsilon^\ell \cdot \nabla \delta w_\epsilon + 2\eta \delta D_\epsilon \delta w_\epsilon \nabla w_\epsilon^\ell \cdot \nabla \eta dx ds \\
\le & C\epsilon^2 \|\delta D_\epsilon\|_{L^\infty((0,t)\times\Omega_\epsilon^h)} \|\nabla w_\epsilon^\ell\|_{L^2((0,t)\times\Omega_\epsilon^h)} \|\eta\nabla\delta w_\epsilon\|_{L^2((0,t)\times\Omega_\epsilon^h)} \\
&+ C\epsilon^2 \|\eta\delta w_\epsilon\|_{L^2((0,t)\times\Omega_\epsilon^h)} \|\delta D_\epsilon\|_{L^\infty((0,t)\times\Omega_\epsilon^h)} \|\nabla w_\epsilon^\ell\|_{L^2((0,t)\times\Omega_\epsilon^h)} \\
\le & C(\theta)|\ell\epsilon|^2 + \theta\epsilon^2 \|\eta\nabla\delta w_\epsilon\|^2_{L^2((0,t)\times\Omega_\epsilon^h)} + C\|\eta\delta w_\epsilon\|^2_{L^2((0,t)\times\Omega_\epsilon^h)} + C\epsilon^2|\ell\epsilon|^2.
\end{aligned}$$

For the term $A_\epsilon^{(2)}$ we use Assumptions (A2) and (A3) to conclude that

$$\left\| \frac{D_\epsilon}{J_\epsilon} \nabla J_\epsilon \right\|_{L^\infty((0,T)\times\Omega_\epsilon^h)} \le \frac{C}{\epsilon}.$$

Using the above gives

$$\begin{aligned}
\int_0^t A_\epsilon^{(2)} ds = &\, 2\epsilon^2 \int_0^t \int_{\Omega_\epsilon^h} \delta w_\epsilon^2 \frac{D_\epsilon}{J_\epsilon} \nabla J_\epsilon \cdot \nabla \eta \eta + \delta w_\epsilon \frac{D_\epsilon}{J_\epsilon} \nabla J_\epsilon \cdot \nabla \delta w_\epsilon \eta^2 dx ds \\
\le &\, C\epsilon \|\delta w_\epsilon\|^2_{L^2((0,t)\times\Omega_\epsilon^h)} + C\epsilon \|\eta\delta w_\epsilon\|_{L^2((0,t)\times\Omega_\epsilon^h)} \|\eta\nabla\delta w_\epsilon\|_{L^2((0,t)\times\Omega_\epsilon^h)} \\
\le &\, C\epsilon + C(\theta)\|\eta\delta w_\epsilon\|^2_{L^2((0,t)\times\Omega_\epsilon^h)} + \theta\epsilon^2\|\eta\nabla\delta w_\epsilon\|^2_{L^2((0,t)\times\Omega_\epsilon^h)}.
\end{aligned}$$

For $A_\epsilon^{(3)}$, we use Assumption (A4) to obtain

$$\left\| \delta \left(\frac{D_\epsilon}{J_\epsilon} \nabla J_\epsilon \right) \right\|_{L^\infty((0,T)\times\Omega_\epsilon^h)} \le C|l|.$$

This implies that

$$\begin{aligned}
\int_0^t A_\epsilon^{(3)} ds \le & C|\ell\epsilon|\epsilon \|w_\epsilon^\ell\|_{L^2((0,t)\times\Omega_\epsilon^h)} \|\eta\nabla\delta w_\epsilon\|_{L^2((0,t)\times\Omega_\epsilon^h)} \\
&+ C|\ell\epsilon|\epsilon \|\eta\delta w_\epsilon\|_{L^2((0,t)\times\Omega_\epsilon^h)} \|w_\epsilon^\ell\|_{L^2((0,t)\times\Omega_\epsilon^h)} \\
\le & C(\theta)|\ell\epsilon|^2 + \theta\epsilon^2 \|\eta\nabla\delta w_\epsilon\|^2_{L^2((0,t)\times\Omega_\epsilon^h)} + C\|\eta\delta w_\epsilon\|^2_{L^2((0,t)\times\Omega_\epsilon^h)} + C|\ell\epsilon|^2.
\end{aligned}$$

Using the assumptions on $\tilde{q}_\epsilon$, we obtain

$$\begin{aligned}
\int_0^t A_\epsilon^{(4)} ds \le & C\epsilon \|w_\epsilon^\ell\|_{L^2((0,t)\times\Omega_\epsilon^h)} \|\delta q_\epsilon\|_{L^\infty((0,t)\times\Omega_\epsilon^h)} \|\eta\nabla w_\epsilon\|_{L^2(\Omega_\epsilon^h)} \\
&+ C\epsilon \|w_\epsilon^\ell\|_{L^2((0,t)\times\Omega_\epsilon^h)} \|\eta\delta w_\epsilon\|_{L^2((0,t)\times\Omega_\epsilon^h)} \|\delta q_\epsilon\|_{L^\infty((0,t)\times\Omega_\epsilon^h)} \\
\le & C(\theta)|\ell\epsilon|^2 + \theta\epsilon^2 \|\eta\nabla\delta w_\epsilon\|^2_{L^2((0,t)\times\Omega_\epsilon^h)} + C\|\eta\delta w_\epsilon\|^2_{L^2((0,t)\times\Omega_\epsilon^h)} + C\epsilon^2|\ell\epsilon|^2.
\end{aligned}$$

For the fifth term, employing similar arguments as above leads to

$$\int_0^t A_\epsilon^{(5)} ds \le C(\theta)\|\eta\delta w_\epsilon\|^2_{L^2((0,t)\times\Omega_\epsilon^h)} + \theta\epsilon^2\|\eta\nabla\delta w_\epsilon\|^2_{L^2((0,t)\times\Omega_\epsilon^h)} + C\epsilon.$$

For $A_\epsilon^{(6)}$, we use Assumption (A5) to obtain

$$\begin{aligned}\int_0^t A_\epsilon^{(6)} ds \le& \|\delta v_\epsilon\|_{L^\infty((0,t)\times\Omega_\epsilon^h)}\|w_\epsilon^\ell\|_{L^2((0,t)\times\Omega_\epsilon^h)}\|\eta\nabla\delta w_\epsilon\|_{L^2((0,t)\times\Omega_\epsilon^h)}\\ &+ C\|w_\epsilon^\ell\|_{L^2((0,t)\times\Omega_\epsilon^h)}\|\eta\delta w_\epsilon\|_{L^2((0,t)\times\Omega_\epsilon^h)}\|\delta v_\epsilon\|_{L^\infty((0,t)\times\Omega_\epsilon^h)}\\ \le& C(\theta)|\ell\epsilon|^2 + \theta\epsilon^2\|\eta\nabla\delta w_\epsilon\|^2_{L^2((0,t)\times\Omega_\epsilon^h)} + C\|\eta\delta w_\epsilon\|^2_{L^2((0,t)\times\Omega_\epsilon^h)} + C\epsilon^2|\ell\epsilon|^2,\end{aligned}$$

while for $A_\epsilon^{(7)}$ we get

$$\int_0^t A_\epsilon^{(7)} ds \le C(\theta)\|\eta\delta w_\epsilon\|^2_{L^2((0,t)\times\Omega_\epsilon^h)} + \theta\epsilon^2\|\eta\nabla\delta w_\epsilon\|^2_{L^2((0,t)\times\Omega_\epsilon^h)} + C\epsilon.$$

From the nonlinear terms, we only consider the boundary term $A_\epsilon^{(9)}$, since the other one can be treated in a similar way. One has

$$\begin{aligned}\int_0^t A_\epsilon^{(9)} ds =& \epsilon\int_0^t\int_{\Gamma_\epsilon^h} \delta J_\epsilon g(u_\epsilon^\ell)\eta^2\delta w_\epsilon d\sigma ds + \epsilon\int_0^t\int_{\Gamma_\epsilon^h} J_\epsilon\big(g(u_\epsilon^\ell) - g(u_\epsilon)\big)\eta^2\delta w_\epsilon d\sigma ds\\ =&: B_\epsilon^{(1)} + B_\epsilon^{(2)}\end{aligned}$$

Using Assumption (A4), Lemma 1, and the trace inequality (7), for $B_\epsilon^{(1)}$ we get

$$\begin{aligned}B_\epsilon^1 \le& C\epsilon^{\frac{1}{2}}|\ell\epsilon|\|\eta\delta w_\epsilon\|_{L^2((0,t)\times\Gamma_\epsilon^h)} + C\epsilon|\ell\epsilon|\|u_\epsilon\|_{L^2((0,t)\times\Gamma_\epsilon^h)}\|\eta\delta w_\epsilon\|_{L^2((0,t)\times\Gamma_\epsilon^h)}\\ \le& C\epsilon^{\frac{1}{2}}|\ell\epsilon|\|\eta\delta w_\epsilon\|_{L^2((0,t)\times\Gamma_\epsilon^h)}\\ \le& C(\theta)|\ell\epsilon|\|\eta\delta w_\epsilon\|_{L^2((0,t)\times\Omega_\epsilon^h)} + \theta\epsilon|\ell\epsilon|\|\nabla(\eta\delta w_\epsilon)\|_{L^2((0,t)\times\Omega_\epsilon^h)}\\ \le& C(\theta)\|\eta\delta w_\epsilon\|^2_{L^2((0,t)\times\Omega_\epsilon^h)} + C(\theta)|\ell\epsilon|^2 + C\epsilon|\ell\epsilon| + \theta\epsilon^2\|\eta\nabla\delta w_\epsilon\|^2_{L^2((0,t)\times\Omega_\epsilon^h)}.\end{aligned}$$

Using the positivity of J_ϵ and similar arguments as for $B_\epsilon^{(1)}$ gives

$$\begin{aligned}B_\epsilon^{(2)} \le& C\epsilon\int_0^t\int_{\Gamma_\epsilon^h} |J_\epsilon u_\epsilon^\ell - J_\epsilon u_\epsilon|\eta^2|\delta w_\epsilon| d\sigma ds\\ \le& C\epsilon\int_0^t\int_{\Gamma_\epsilon^h} |\delta J_\epsilon||u_\epsilon^\ell|\eta^2|\delta w_\epsilon| + \eta^2|\delta w_\epsilon|^2 d\sigma ds\\ \le& C|\ell\epsilon|\epsilon\|u_\epsilon^\ell\|_{L^2((0,t)\times\Gamma_\epsilon^h)}\|\eta\delta w_\epsilon\|_{L^2((0,t)\times\Gamma_\epsilon^h)} + C\epsilon\|\eta\delta w_\epsilon\|^2_{L^2((0,t)\times\Gamma_\epsilon^h)}\\ \le& C(\theta)\|\eta\delta w_\epsilon\|^2_{L^2((0,t)\times\Omega_\epsilon^h)} + C(\theta)|\ell\epsilon|^2 + C\epsilon|\ell\epsilon| + \theta\epsilon^2\|\eta\nabla\delta w_\epsilon\|_{L^2((0,t)\times\Omega_\epsilon^h)} + C\epsilon^2.\end{aligned}$$

Similarly, using Assumption (A5), we obtain

$$\begin{aligned}\int_0^t A_\epsilon^{(10)} ds =& \int_0^t\int_{\Gamma_\epsilon^h} \delta w_\epsilon^2 v_\epsilon^\ell\cdot\nu\eta^2 d\sigma dx + \int_0^t\int_{\Gamma_\epsilon^h} w_\epsilon\big[v_\epsilon^\ell - v_\epsilon\big]\cdot\nu\eta^2\delta w_\epsilon d\sigma ds\\ \le& C\epsilon\|\eta\delta w_\epsilon\|^2_{L^2((0,t)\times\Gamma_\epsilon^h)} + C\epsilon^2|l|\int_0^t\int_{\Gamma_\epsilon^h} |w_\epsilon|\eta^2|\delta w_\epsilon| d\sigma ds\\ \le& C(\theta)\|\eta\delta w_\epsilon\|^2_{L^2((0,t)\times\Omega_\epsilon^h)} + \theta\epsilon^2\|\eta\nabla\delta w_\epsilon\|^2_{L^2((0,t)\times\Omega_\epsilon^h)} + C\epsilon^2 + C|\epsilon l|^2.\end{aligned}$$

Integrating (8) with respect to time, applying the above estimates and the Gronwall-inequality, after choosing θ small enough gives the desired result. □

4 Two-scale compactness results

In this section we prove a general strong two-scale compactness result for an arbitrary sequence based on *a priori* estimates and estimates for the differences of the shifts. We make use of the unfolding operator in perforated domains, see [8]. First of all, let us repeat the definition of two-scale convergence, which was introduced in [20] and [2].

Definition 2. *A sequence $u_\epsilon \in L^p((0,T)\times\Omega)$ for $p \in [1,\infty)$ is said to converge in the two-scale sense to the limit function $u_0 \in L^p((0,T)\times\Omega\times Y)$, if for every $\phi \in C\left([0,T]\times\overline{\Omega}, C_{per}\left(\overline{Y}\right)\right)$ the following relation holds*

$$\lim_{\epsilon\to 0}\int_0^T\int_\Omega u_\epsilon(t,x)\phi\left(t,x,\frac{x}{\epsilon}\right)dxdt = \int_0^T\int_\Omega\int_Y u_0(t,x,y)\phi(t,x,y)dydxdt.$$

A two-scale convergent sequence u_ϵ convergences strongly in the two-scale sense to u_0, if

$$\lim_{\epsilon\to 0}\|u_\epsilon\|_{L^p((0,T)\times\Omega)} = \|u_0\|_{L^p((0,T)\times\Omega\times Y)}.$$

Remark 3. *By replacing Y with Y^* and Ω with Ω_ϵ, the Definition 2 can be easily extended to sequences in $L^p((0,T)\times\Omega_\epsilon)$. In both cases we will use the same notation for the two-scale convergence.*

In [3, 18] the method of two-scale convergence was extended to oscillating surfaces:

Definition 3. *A sequence of functions $u_\epsilon \in L^p((0,T)\times\Gamma_\epsilon)$ for $p \in [1,\infty)$ is said to converge in the two-scale sense on the surface Γ_ϵ to a limit $u_0 \in L^p((0,T)\times\Omega\times\Gamma)$, if for every $\phi \in C\left([0,T]\times\overline{\Omega}, C_{per}(\Gamma)\right)$ it holds that*

$$\lim_{\epsilon\to 0}\epsilon\int_0^T\int_{\Gamma_\epsilon} u_\epsilon(t,x)\phi\left(t,x,\frac{x}{\epsilon}\right)d\sigma dt = \int_0^T\int_\Omega\int_\Gamma u_0(t,x,y)\phi(t,x,y)d\sigma_y dxdt.$$

We say a two-scale convergent sequence u_ϵ converges strongly in the two-scale sense, if additionally it holds that

$$\lim_{\epsilon\to 0}\epsilon^{\frac{1}{p}}\|u_\epsilon\|_{L^2((0,T)\times\Gamma_\epsilon)} = \|u_0\|_{L^p((0,T)\times\Omega\times\Gamma)}.$$

An equivalent characterization of the two-scale convergence can be given via the unfolding operator, see [8], which is defined by ($p \in [1,\infty)$) for perforated domains by

$$\mathcal{T}_\epsilon : L^p((0,T)\times\Omega_\epsilon) \to L^p((0,T)\times\Omega\times Y^*), \quad \mathcal{T}_\epsilon u_\epsilon(t,x,y) = u_\epsilon\left(t, \epsilon\left[\frac{x}{\epsilon}\right] + \epsilon y\right),$$

where $[\cdot]$ denotes the integer part. In the same way we can define the unfolding operator on the whole domin Ω and we use the same notation. We also define the boundary unfolding operator by

$$\mathcal{T}_\epsilon^b : L^p((0,T)\times\Gamma_\epsilon) \to L^p((0,T)\times\Omega\times\Gamma), \quad \mathcal{T}_\epsilon^b u_\epsilon(t,x,y) = u_\epsilon\left(t, \epsilon\left[\frac{x}{\epsilon}\right] + \epsilon y\right).$$

Let us summarize some important properties of the unfolding operator, which can be found in [8]

(i) For $u_\epsilon, v_\epsilon \in L^2((0,T)\times\Omega_\epsilon)$ it holds that

$$(u_\epsilon, v_\epsilon)_{L^2((0,T)\times\Omega_\epsilon)} = (\mathcal{T}_\epsilon u_\epsilon, \mathcal{T}_\epsilon v_\epsilon)_{L^2((0,T)\times\Omega\times Y^*)}.$$

(ii) For $u_\epsilon \in L^2((0,T), H^1(\Omega_\epsilon))$ we have $\nabla_y \mathcal{T}_\epsilon u_\epsilon = \epsilon \mathcal{T}_\epsilon(\nabla u_\epsilon)$.

(iii) For $u_\epsilon, v_\epsilon \in L^2((0,T)\times\Gamma_\epsilon)$ it holds that

$$(u_\epsilon, v_\epsilon)_{L^2((0,T)\times\Gamma_\epsilon)} = \epsilon(\mathcal{T}^b_\epsilon u_\epsilon, \mathcal{T}^b_\epsilon v_\epsilon)_{L^2((0,T)\times\Omega\times\Gamma)}.$$

The statements above remain valid if we replace Ω_ϵ by Ω.

We have the following relation between the unfolding operator and the two-scale convergence, see [5].

Lemma 3. *For every bounded sequence $u_\epsilon \in L^p((0,T)\times\Omega)$ the following statements are equivalent*

(i) $u_\epsilon \to u_0$ weakly/strongly in the two-scale sense for $u_0 \in L^p((0,T)\times\Omega\times Y)$.

(ii) $\mathcal{T}_\epsilon u_\epsilon \to u_0$ weakly/strongly in $L^p((0,T)\times\Omega\times Y)$.

The same result holds for u_ϵ in $L^p((0,T)\times\Gamma_\epsilon)$ with $\epsilon^{\frac{1}{p}}\|u_\epsilon\|_{L^p((0,T)\times\Gamma_\epsilon)} \le C$ and the boundary unfolding operator $\mathcal{T}^b_\epsilon$. The result is also true, if we replace Ω with Ω_ϵ and Y with Y^*.

Now, we define the averaging operator $\mathcal{U}_\epsilon : L^2((0,T)\times\Omega\times Y^*) \to L^2((0,T)\times\Omega_\epsilon)$ as the L^2-adjoint of the unfolding operator. An elemental calculation, see [8], shows

$$\mathcal{U}_\epsilon(\phi)(t,x) = \int_{Y^*} \phi\left(t, \epsilon\left(y + \left[\frac{x}{\epsilon}\right]\right), \left\{\frac{x}{\epsilon}\right\}\right) dy,$$

with $\{z\} = z - [z]$. As an immediate consequence of the L^2-adjointness we obtain

$$\|\mathcal{U}_\epsilon(\phi)\|_{L^2((0,T)\times\Omega_\epsilon)} \le \|\phi\|_{L^2((0,T)\times\Omega\times Y^*)}. \tag{9}$$

Further, let us define the space

$$\mathcal{H}^1_0(Y^*) := \left\{\phi \in H^1(Y^*) \,:\, \phi = 0 \text{ on } \partial Y \cap \partial Y^*\right\}.$$

Then we have the following result

Proposition 2. *For all $\phi \in L^2((0,T)\times\Omega, \mathcal{H}^1_0(Y^*))$ it holds that $\epsilon\nabla_x \mathcal{U}_\epsilon(\phi) = \mathcal{U}_\epsilon(\nabla_y \phi)$. Especially, we have*

$$\|\mathcal{U}_\epsilon(\phi)\|_{L^2((0,T),\mathcal{H}_\epsilon)} \le \|\phi\|_{L^2((0,T)\times\Omega,\mathcal{H}^1_0(Y^*))}.$$

In other words, the restriction of $\mathcal{U}_\epsilon$ to $L^2((0,T)\times\Omega, \mathcal{H}^1_0(Y^*))$ fulfills

$$\mathcal{U}_\epsilon : L^2((0,T)\times\Omega, \mathcal{H}^1_0(Y^*)) \to L^2((0,T), \mathcal{H}_\epsilon).$$

Proof. This is an easy consequence of the L^2-adjointness of $\mathcal{U}_\epsilon$ and $\nabla_y \mathcal{T}_\epsilon = \epsilon \mathcal{T}_\epsilon \nabla_x$. In fact for $\phi \in L^2((0,T)\times\Omega, \mathcal{H}_0^1(Y^*))$ and $\psi \in C_0^\infty(\Omega_\epsilon)$ we have almost everywhere in $(0,T)$

$$\begin{aligned}\int_{\Omega_\epsilon} \mathcal{U}_\epsilon(\phi)\partial_{x_i}\psi dx &= \int_\Omega \int_{Y^*} \phi \mathcal{T}_\epsilon(\partial_{x_i}\psi) dy dx \\ &= \frac{1}{\epsilon}\int_\Omega \int_{Y^*} \phi \partial_{y_i}\mathcal{T}_\epsilon(\psi) dy dx \\ &= -\frac{1}{\epsilon}\int_\Omega \int_{Y^*} \partial_{y_i}\phi \mathcal{T}_\epsilon(\psi) dy dx + \frac{1}{\epsilon}\int_\Omega \int_{\partial Y^*} \phi \mathcal{T}_\epsilon(\psi)\nu_i d\sigma_y dx \\ &= -\frac{1}{\epsilon}\int_{\Omega_\epsilon} \mathcal{U}_\epsilon(\partial_{y_i}\phi)\psi dx,\end{aligned}$$

where the boundary term vanishes since $\mathcal{T}_\epsilon(\psi) = 0$ on Γ and $\phi = 0$ on $\partial Y \cap \partial Y^*$. The inequality is an easy consequence of (9) and the definition of $\mathcal{H}_\epsilon$. □

This result gives a formula for the generalized time-derivative of $\mathcal{T}_\epsilon u_\epsilon$.

Proposition 3. *Let $v_\epsilon \in L^2((0,T)\times\Omega_\epsilon) \cap H^1((0,T),\mathcal{H}_\epsilon')$. Then we have*

$$\mathcal{T}_\epsilon v_\epsilon \in H^1((0,T), L^2(\Omega, \mathcal{H}_0^1(Y^*))')$$

with

$$\langle \partial_t \mathcal{T}_\epsilon v_\epsilon, \phi\rangle_{L^2(\Omega,\mathcal{H}_0^1(Y^*))', L^2(\Omega,\mathcal{H}_0^1(Y^*))} = \langle \partial_t v_\epsilon, \mathcal{U}_\epsilon(\phi)\rangle_{\Omega_\epsilon}.$$

Additionally, we have

$$\|\partial_t \mathcal{T}_\epsilon v_\epsilon\|_{L^2((0,T),L^2(\Omega,\mathcal{H}_0^1(Y^*))')} \leq \|\partial_t v_\epsilon\|_{L^2((0,T),\mathcal{H}_\epsilon')}.$$

Proof. For $\phi \in L^2(\Omega, \mathcal{H}_0^1(Y^*))$ and $\psi \in \mathcal{D}(0,T)$ we obtain immediately from Proposition 2

$$\begin{aligned}&\int_0^T \int_\Omega \int_{Y^*} \mathcal{T}_\epsilon v_\epsilon(t,x,y)\phi(x,y)\psi'(t) dy dx dt \\ &\quad = \int_0^T \int_{\Omega_\epsilon} v_\epsilon(t,x)\mathcal{U}_\epsilon(\phi)(x)\psi'(t) dx dt = -\int_0^T \langle \partial_t v_\epsilon(t), \mathcal{U}_\epsilon(\phi)\rangle_{\Omega_\epsilon}\psi(t) dt.\end{aligned}$$

The inequality is a direct consequence of $\langle \partial_t v_\epsilon, \mathcal{U}_\epsilon(\phi)\rangle_{\Omega_\epsilon} = \langle \partial_t v_\epsilon, \mathcal{U}_\epsilon(\phi)\rangle_{\mathcal{H}_\epsilon',\mathcal{H}_\epsilon}$ and Proposition 2 . □

In the next proposition we show that, with under suitable a priori estimates, the existence of a generalized time-derivative for a two-scale convergent sequence extends to the limit function. First, we introduce the difference quotient with respect to time. Given a Banach space X and with $0 < h \ll 1$, for an arbitrary function $v \in L^2((0,T),X)$ we define

$$\partial_t^h v(t,x) := \frac{v(t+h,x) - v(t,x)}{h} \quad \text{for } t \in (0, T-h),\ x \in X.$$

Proposition 4. *Let $v_\epsilon \in L^2((0,T),\mathcal{H}_\epsilon) \cap H^1((0,T),\mathcal{H}_\epsilon')$ with*

$$\|v_\epsilon\|_{L^2((0,T),\mathcal{H}_\epsilon)} + \|\partial_t v_\epsilon\|_{L^2((0,T),\mathcal{H}_\epsilon')} \leq C.$$

Then there exists $v_0 \in L^2((0,T)\times\Omega, H^1_{\text{per}}(Y^*)) \cap H^1((0,T), L^2(\Omega, H^1_{\text{per}}(Y^*)'))$, such that up to a subsequence it holds that

$$\begin{aligned} v_\epsilon &\to v_0 && \textit{in the two-scale sense,} \\ \epsilon\nabla v_\epsilon &\to \nabla_y v_0 && \textit{in the two-scale sense.} \end{aligned}$$

Further, for every $\phi \in \mathcal{D}\big((0,T), L^2(\Omega), H^1_{\text{per}}(Y^*)\big)$ and $\phi_\epsilon(t,x) := \phi\left(t,x,\frac{x}{\epsilon}\right)$ it holds that (up to a subsequence)

$$\lim_{\epsilon\to 0}\int_0^T \langle \partial_t v_\epsilon, \phi_\epsilon\rangle_{H^1(\Omega_\epsilon)',H^1(\Omega_\epsilon)} dt = \int_0^T \langle \partial_t v_0, \phi\rangle_{L^2(\Omega,H^1_{\text{per}}(Y^*)'),L^2(\Omega,H^1_{\text{per}}(Y^*))} dt.$$

Proof. By standard two-scale compactness results, a function $v_0 \in L^2((0,T)\times\Omega, H^1_{\text{per}}(Y^*))$ exists such that (up to a subsequence)

$$\begin{aligned} v_\epsilon &\to v_0 && \text{in the two-scale sense,} \\ \epsilon\nabla v_\epsilon &\to \nabla_y v_0 && \text{in the two-scale sense.} \end{aligned}$$

To establish the existence of the weak time-derivative of v_0 we show that $\partial_t^h v_0$ is bounded in $L^2((0,T-h), L^2(\Omega, H^1_{\text{per}}(Y^*)'))$ uniformly with respect to h. For all $\phi \in L^2((0,T-h), L^2(\Omega, H^1_{\text{per}}(Y^*)))$ with

$$\|\phi\|_{L^2((0,T-h),L^2(\Omega,H^1_{\text{per}}(Y^*)))} \leq 1$$

and $\phi_\epsilon(t,x) := \phi\left(t,x,\frac{x}{\epsilon}\right)$ it holds that

$$\begin{aligned} &\langle \partial_t^h v_0, \phi\rangle_{L^2((0,T-h),L^2(\Omega,H^1_{\text{per}}(Y^*)')),L^2((0,T-h),L^2(\Omega,H^1_{\text{per}}(Y^*)))} \\ &\quad= \int_0^{T-h}\int_\Omega\int_{Y^*} \partial_t^h v_0(t,x,y)\phi(t,x,y)dydxdt \\ &\quad= \lim_{\epsilon\to 0}\int_0^{T-h}\int_{\Omega_\epsilon} \partial_t^h v_\epsilon(t,x)\phi_\epsilon(t,x)dxdt \\ &\quad= \lim_{\epsilon\to 0}\int_0^{T-h} \langle \partial_t^h v_\epsilon(t), \phi_\epsilon(t,\cdot)\rangle_{\mathcal{H}_\epsilon',\mathcal{H}_\epsilon} dt \\ &\quad\leq \lim_{\epsilon} \|\partial_t^h v_\epsilon\|_{L^2((0,T-h),\mathcal{H}_\epsilon')}\|\phi_\epsilon\|_{L^2((0,T-h),\mathcal{H}_\epsilon)} \\ &\quad\leq C\lim_{\epsilon\to 0}\|\partial_t v_\epsilon\|_{L^2((0,T),\mathcal{H}_\epsilon')} \leq C. \end{aligned}$$

This implies

$$\|\partial_t^h v_0\|_{L^2((0,T),L^2(\Omega,H^1_{\text{per}}(Y^*)'))} \leq C$$

uniformly with respect to h. Hence, $\partial_t v_0 \in L^2((0,T), L^2(\Omega, H^1_{\text{per}}(Y^*)'))$. Further, for $\phi \in \mathcal{D}\big((0,T)\times\overline{\Omega}, C^\infty_{\text{per}}(Y^*)\big)$ and $\phi_\epsilon(t,x) := \phi\left(t,x,\frac{x}{\epsilon}\right)$ we obtain by integration by parts

$$\begin{aligned}\int_0^T \langle \partial_t v_\epsilon, \phi_\epsilon \rangle_{H^1(\Omega_\epsilon)', H^1(\Omega_\epsilon)} dt &= -\int_0^T \int_{\Omega_\epsilon} v_\epsilon(t,x) \partial_t \phi\left(t, x, \frac{x}{\epsilon}\right) dxdt \\ &\overset{\epsilon\to 0}{\longrightarrow} -\int_0^T \int_\Omega \int_{Y^*} v_0(t,x,y)\partial_t \phi(t,x,y) dydxdt \\ &= \int_0^T \langle \partial_t v_0, \phi \rangle_{L^2(\Omega, H^1_{\mathrm{per}}(Y^*)'), L^2(\Omega, H^1_{\mathrm{per}}(Y^*))}.\end{aligned}$$

The desired result follows by density arguments. □

Remark 4. *If the condition for the time-derivative $\partial_t v_\epsilon$ in Proposition 4 is replaced by the weaker one, namely $\partial_t v_\epsilon \in L^2((0,T), H^1(\Omega_\epsilon)')$ with*

$$\|\partial_t v_\epsilon\|_{L^2((0,T),H^1(\Omega_\epsilon)')} \leq C,$$

the existence of $\partial_t v_0$ is not guaranteed anymore. However, using similar arguments as in the proof above, one shows that the time-derivative of $\bar{v}_0 := \int_{Y^*} v_0 dy$ exists. More precisely, we have $\bar{v}_0 \in L^2((0,T), H^1(\Omega)')$ and for all $\phi \in H^1(\Omega)$ it holds that

$$\lim_{\epsilon\to 0} \int_0^T \langle \partial_t v_\epsilon, \phi \rangle_{H^1(\Omega_\epsilon)', H^1(\Omega_\epsilon)} = \int_0^T \langle \partial_t \bar{v}_0, \phi \rangle_{H^1(\Omega)', H^1(\Omega)}.$$

For the proof of the strong two-scale compactness result we make use of the following lemma, which gives a relation between differences of shifted unfolded functions and the functions itself.

Lemma 4. *Let $v_\epsilon \in L^2((0,T) \times \Omega_\epsilon)$. For $0 < h \ll 1$ and $|z| < h$ it holds that*

$$\left\|\mathcal{T}_\epsilon v_\epsilon(t, x+z, y) - \mathcal{T}_\epsilon v_\epsilon\right\|^2_{L^2(0,T)\times\Omega_{2h}\times Y^*)} \leq \sum_{j\in\{0,1\}^n} \|\delta_l v_\epsilon\|^2_{L^2((0,T)\times\Omega^h_\epsilon)},$$

with $l = l(\epsilon, z, j) = j + \left[\frac{z}{\epsilon}\right]$.

Proof. This result was proved for thin domains in [19, p. 709-710] and extended to domains in [12, Proof of Theorem 3]. □

Now, we are able to formulate the strong two-scale compactness result.

Theorem 1. *Consider a sequence of functions $v_\epsilon \in L^2((0,T), H^1(\Omega_\epsilon)) \cap H^1((0,T), \mathcal{H}'_\epsilon)$ satisfying the following conditions*

(i) An ϵ-independent $C > 0$ exists such that

$$\|v_\epsilon\|_{L^2((0,T),\mathcal{H}_\epsilon)} + \|\partial_t v_\epsilon\|_{L^2((0,T),\mathcal{H}'_\epsilon)} \leq C.$$

(ii) For $0 < h \ll 1$ and $\ell \in \mathbb{Z}^n$ with $|\ell\epsilon| < h$, it holds that

$$\|\delta v_\epsilon\|_{L^2((0,T),L^2(\Omega^h_\epsilon))} + \epsilon\|\nabla \delta v_\epsilon\|_{L^2((0,T),L^2(\Omega^h_\epsilon))} \overset{\epsilon l\to 0}{\longrightarrow} 0.$$

Then, there exists $v_0 \in L^2((0,T) \times \Omega, H^1_{\mathrm{per}}(Y^*))$, such that for $\beta \in \left(\frac{1}{2}, 1\right)$ and $p \in [1,2)$ it holds that

$$\mathcal{T}_\epsilon v_\epsilon \to v_0 \quad \text{in } L^p(\Omega, L^2((0,T), H^\beta(Y^*))).$$

Remark 5. *If we replace the condition (ii) by*

$$\|\delta v_\epsilon\|_{L^2((0,T),L^2(\Omega_\epsilon^h))} + \epsilon\|\nabla\delta v_\epsilon\|_{L^2((0,T),L^2(\Omega_\epsilon^h))} \overset{h\to 0}{\longrightarrow} 0,$$

and $|\ell\epsilon| < h$, we obtain the strong convergence of $\mathcal{T}_\epsilon v_\epsilon$ to v_0 in $L^2((0,T)\times\Omega, H^\beta(Y))$. Therefore, in the proof below we have to make use of [11, Theorem 2.2].

Proof of Theorem 1. We apply [11, Corollary 2.5] , which gives a generalization of Simon compactness results from [28] for domains in $\mathbb{R}^n$, and therefore we have to check the following properties:

(a) For $A \subset \Omega$ measurable, the sequence $V_\epsilon(t,y) := \int_A \mathcal{T}_\epsilon v_\epsilon(t,x,y)dx$ is relatively compact in $L^2((0,T),H^\beta(Y^*))$.

(b) For $0 < h \ll 1$ and $|z| < h$ it holds that

$$\sup_\epsilon \big\|\mathcal{T}_\epsilon v_\epsilon(t,x+z,y) - \mathcal{T}_\epsilon v_\epsilon\big\|_{L^p(\Omega^h,L^2((0,T),H^\beta(Y^*)))} \overset{z\to 0}{\longrightarrow} 0.$$

(c) It holds that $(h > 0)$

$$\sup_\epsilon \big\|\mathcal{T}_\epsilon v_\epsilon\big\|_{L^p(\Omega\setminus\Omega^h,L^2((0,T),H^\beta(Y^*)))} \overset{h\to 0}{\longrightarrow} 0.$$

First of all, let $A \subset \Omega$ and V_ϵ defined as in (a). The properties of the unfolding operator and the condition (i) imply

$$\|V_\epsilon\|^2_{L^2((0,T),H^1(Y^*))} \le \big\|\mathcal{T}_\epsilon v_\epsilon\big\|^2_{L^2((0,T)\times\Omega,H^1(Y^*))} \le C\|v_\epsilon\|^2_{L^2((0,T),\mathcal{H}_\epsilon)} \le C.$$

Further, it holds for all $\phi \in \mathcal{H}_0^1(Y^*)$ that

$$\langle\partial_t V_\epsilon,\phi\rangle_{\mathcal{H}_0^1(Y^*)',\mathcal{H}_0^1(Y^*)} = \big\langle\partial_t\mathcal{T}_\epsilon v_\epsilon,\chi_A\phi\big\rangle_{L^2(\Omega,\mathcal{H}_0^1(Y^*))',L^2(\Omega,\mathcal{H}_0^1(Y^*))},$$

where χ_A denotes the characteristic function on A. This implies for $\phi \in \mathcal{H}_0^1(Y^*)$ with $\|\phi\|_{H^1(Y^*)} \le 1$ together with Proposition 3

$$\big|\langle\partial_t V_\epsilon,\phi\rangle_{\mathcal{H}_0^1(Y^*)',\mathcal{H}_0^1(Y^*)}\big| \le \big\|\partial_t\mathcal{T}_\epsilon v_\epsilon\big\|_{L^2(\Omega,\mathcal{H}_0^1(Y^*))'}\|\chi_A\phi\|_{L^2(\Omega,\mathcal{H}_0^1(Y^*))} \le C\|\partial_t v_\epsilon\|_{\mathcal{H}_\epsilon}.$$

Now, the condition (i) implies that

$$\|\partial_t V_\epsilon\|_{L^2((0,T),\mathcal{H}_0^1(Y^*))'} \le C,$$

i. e. V_ϵ is bounded in $L^2((0,T),H^1(Y^*))\cap H^1((0,T),\mathcal{H}_0^1(Y^*)')$. Since the embedding $H^1(Y^*) \hookrightarrow H^\beta(Y^*)$ is compact for $\beta \in \left(\frac{1}{2},1\right)$ whereas $H^\beta(Y^*) \hookrightarrow \mathcal{H}_0^1(Y^*)'$ continuously, the Aubin-Lions Lemma implies that V_ϵ is relatively compact in $L^2((0,T),H^\beta(Y^*))$. This is condition (a) above.

To prove (b), we use Lemma 4 to obtain that for $0 < h \ll 1$ and $|z| < h$ and ϵ small enough

$$\begin{aligned}
&\big\|\mathcal{T}_\epsilon v_\epsilon(t,x+z,y) - \mathcal{T}_\epsilon v_\epsilon\big\|^2_{L^p(\Omega_{2h},L^2((0,T),H^\beta(Y^*)))}\\
&\quad\le C\big\|\mathcal{T}_\epsilon v_\epsilon(t,x+z,y) - \mathcal{T}_\epsilon v_\epsilon\big\|^2_{L^2((0,T)\times\Omega_{2h},H^1(Z))}\\
&\quad\le \sum_{j\in\{0,1\}^n} \|\delta_l v_\epsilon\|^2_{L^2((0,T)\times\Omega_\epsilon^h)} + \epsilon^2\|\nabla\delta_l v_\epsilon\|^2_{L^2((0,T)\times\Omega_\epsilon^h)}
\end{aligned}$$

with $l = \epsilon\left(j + \left[\frac{z}{\epsilon}\right]\right)$. Hence, for $\epsilon, z \to 0$ we have $\ell\epsilon \to 0$. Due to the condition (ii) the right-hand side converges to 0 for $\epsilon, z \to 0$. This means, that (b) holds for all but finitely many ϵ. However, for these finitely many ϵ, we can use the standard Kolmogorov compactness result, and therefore (b) is proved.

Condition (c) is an easy consequence of the Hölder inequality. For $p^* := \frac{2p}{2-p}$,

$$\|\mathcal{T}_\epsilon v_\epsilon\|_{L^p(\Omega\setminus\Omega^h, L^2((0,T),H^\beta(Y^*)))} \le \left|\Omega \setminus \Omega^h\right|^{p^*} \|\mathcal{T}_\epsilon v_\epsilon\|_{L^2((0,T)\times\Omega^h, H^1(Z))} \le Ch^{p^*} \overset{h\to 0}{\longrightarrow} 0,$$

where in the last inequality we used the condition (i). □

5 Derivation of the macroscopic model

In this section we use the compactnes results obtained in Section 4 and the *a priori* estimates from Section 3 to derive the macroscopic model, obtained for $\epsilon \to 0$. In a first step, we derive an effective model defined on the reference domain, more precisely the reference element. Eventually, we transform the model to the one defined on a moving domain, described with the help of time- and space-dependent reference elements.

Proposition 5. *Let u_ϵ be a sequence of weak solutions to Problem P_T in (3). There exist a $u_0 \in L^2((0,T)\times\Omega, H^1_{\text{per}}(Y^*))) \cap L^\infty((0,T), L^2(\Omega\times Y^*))$ with $J_0 u_0 \in L^2((0,T)\times\Omega, H^1_{\text{per}}(Y^*)) \cap H^1((0,T), L^2(\Omega, H^1_{\text{per}}(Y^*)'))$, such that, up to a subsequence, for $p \in [1,2)$ and $\beta \in \left(\frac{1}{2}, 1\right)$ it holds that*

$$\begin{aligned}
\mathcal{T}_\epsilon u_\epsilon &\to u_0 && \text{in } L^p(\Omega, L^2((0,T)\times Y^*)),\\
\mathcal{T}_\epsilon u_\epsilon &\to u_0 && \text{in } L^p(\Omega, L^2((0,T)\times \Gamma)),\\
\mathcal{T}_\epsilon (J_\epsilon u_\epsilon) &\to J_0 u_0 && \text{in } L^p(\Omega, L^2((0,T), H^\beta(Y^*))).
\end{aligned}$$

Additionally, for every $\phi \in \mathcal{D}\big((0,T), L^2(\Omega), H^1_{\text{per}}(Y^*)\big)$ and $\phi_\epsilon(t,x) := \phi\left(t, x, \frac{x}{\epsilon}\right)$

$$\lim_{\epsilon\to 0}\int_0^T \langle \partial_t(J_\epsilon u_\epsilon), \phi_\epsilon\rangle_{H^1(\Omega_\epsilon)', H^1(\Omega_\epsilon)} dt = \int_0^T \langle \partial_t(J_0 u_0), \phi\rangle_{L^2(\Omega, H^1_{\text{per}}(Y^*)'), L^2(\Omega, H^1_{\text{per}}(Y^*))} dt.$$

Before giving the proof of Proposition 5, we state briefly a result about the strong convergence of J_ϵ and the regularity of the time-derivative $\partial_t J_0$. This which follows directly from the assumptions on J_ϵ.

Remark 6. *Theorem 1 implies that*

$$\mathcal{T}_\epsilon J_\epsilon \to J_0 \quad \text{in } L^2((0,T)\times\Omega, H^\beta(Y^*))$$

for $\beta \in \left(\frac{1}{2}, 1\right)$. Further, by Proposition 4, $\partial_t J_0 \in L^2((0,T), L^2(\Omega, H^1_{\text{per}}(Y^*)'))$.

Proof of Proposition 5. Assumption (A6) gives $\mathcal{T}_\epsilon J_\epsilon \to J_0$ in $L^q((0,T)\times\Omega\times Y^*)$ for every $q \in [1,\infty)$. The a priori estimates from Lemma 1 imply the existence of a $u_0 \in L^2((0,T)\times\Omega, H^1_{\text{per}}(Y^*))$, such that up to a subsequence

$$\mathcal{T}_\epsilon u_\epsilon \rightharpoonup u_0 \quad \text{weakly in } L^2((0,T)\times\Omega, H^1(Y^*)).$$

Further, since $\mathcal{T}_\epsilon u_\epsilon \in L^\infty((0,T), L^2(\Omega\times Y^*)$ it holds that (see again Lemma 6a)

$$\|u_0\|_{L^p((0,T),L^2(\Omega\times Y^*))} \le \lim_{\epsilon\to 0} \|\mathcal{T}_\epsilon u_\epsilon\|_{L^p((0,T),L^2(\Omega\times Y^*))} \le C$$

for all $p \in [1, \infty)$ with a constant $0 < C$ independent of p. Hence, we have obtained that $u_0 \in L^\infty((0,T), L^2(\Omega \times Y^*))$.

With $\mathcal{T}_\epsilon J_\epsilon \in L^\infty((0,T) \times \Omega \times Y^*)$ and since it converges strongly in the L^q sense for any $q \in [1, \infty)$, we obtain

$$\mathcal{T}_\epsilon(J_\epsilon u_\epsilon) \rightharpoonup J_0 u_0 \quad \text{weakly in } L^2((0,T) \times \Omega \times Y^*).$$

Further, due to Lemmata 1 and 2, recalling Assumption (AD4), the sequence $J_\epsilon u_\epsilon$ fulfills the conditions of Proposition 4 and Theorem 1. Hence $J_0 u_0 \in L^2((0,T) \times \Omega, H^1_{\text{per}}(Y^*)) \cap H^1((0,T), L^2(\Omega, H^1_{\text{per}}(Y^*)'))$ and for $p \in [1,2)$ and $\beta \in \left(\frac{1}{2}, 1\right)$ it holds that, up to a subsequence,

$$\mathcal{T}_\epsilon(J_\epsilon u_\epsilon) \to J_0 u_0 \quad \text{in } L^p(\Omega, L^2((0,T), H^\beta(Y^*))).$$

Moreover, for every $\phi \in \mathcal{D}\big((0,T), L^2(\Omega), H^1_{\text{per}}(Y^*)\big)$ and $\phi_\epsilon(t,x) := \phi\left(t, x, \frac{x}{\epsilon}\right)$,

$$\lim_{\epsilon \to 0} \int_0^T \langle \partial_t(J_\epsilon u_\epsilon), \phi_\epsilon \rangle_{H^1(\Omega_\epsilon)', H^1(\Omega_\epsilon)} dt = \int_0^T \langle \partial_t(J_0 u_0), \phi \rangle_{L^2(\Omega, H^1_{\text{per}}(Y^*)'), L^2(\Omega, H^1_{\text{per}}(Y^*))} dt.$$

Especially, due to the continuity of the embedding $H^\beta(Y^*) \hookrightarrow L^2(\Gamma)$ we obtain

$$\mathcal{T}_\epsilon(J_\epsilon u_\epsilon) \to J_0 u_0 \quad \text{in } L^p(\Omega, L^2((0,T) \times \Gamma)).$$

Now, since $J_\epsilon \geq c > 0$, the strong convergence of $\mathcal{T}_\epsilon J_\epsilon$ (which holds in $L^2((0,T) \times \Omega, H^\beta(Y^*)))$, see Remark 6) and Lebesgue's Dominated Convergence Theorem imply

$$\begin{aligned} \mathcal{T}_\epsilon u_\epsilon &\to u_0 && \text{in } L^p(\Omega, L^2((0,T) \times Y^*)), \\ \mathcal{T}_\epsilon u_\epsilon &\to u_0 && \text{in } L^p(\Omega, L^2((0,T) \times \Gamma)). \end{aligned}$$

□

From the product rule we obtain in the distributional sense

$$\partial_t u_0 = J_0^{-1} \partial_t(J_0 u_0) - J_0^{-1} u_0 \partial_t J_0.$$

The first term on the right-hand side is an element of $L^2((0,T), L^2(\Omega, H^1_{\text{per}}(Y^*)'))$, since $J_0^{-1} \in L^\infty((0,T) \times \Omega, W^{1,\infty}(Y^*))$ and $\partial_t(J_0 u_0) \in L^2((0,T), L^2(\Omega, H^1_{\text{per}}(Y^*)'))$. This is not true for the second term. We only have for $\phi \in L^\infty(\Omega, W^{1,q}_{\text{per}}(Y^*))$ with $q > n$ almost everywhere in $(0,T)$:

$$\begin{aligned} &\langle J_0^{-1} u_0 \partial_t J_0, \phi \rangle_{L^\infty(\Omega, W^{1,q}_{\text{per}}(Y^*))', L^\infty(\Omega, W^{1,q}_{\text{per}}(Y^*))} \\ &\qquad = \langle \partial_t J_0, J_0^{-1} u_0 \phi \rangle_{L^2(\Omega, H^1_{\text{per}}(Y^*)'), L^2(\Omega, H^1_{\text{per}}(Y^*))}, \end{aligned}$$

and therefore $\partial_t u_0 \in L^2((0,T), L^\infty(\Omega, W^{1,q}_{\text{per}}(Y^*))')$.

Corollary 1. *Up to a subsequence, it holds that*

$$\begin{aligned} f(u_\epsilon) &\to f(u_0) && \textit{in the two-scale sense,} \\ g(u_\epsilon) &\to g(u_0) && \textit{in the two-scale sense on } \Gamma_\epsilon. \end{aligned}$$

Proof. This is an easy consequence of the strong convergence results for u_ϵ from Proposition 5. For more details see [14, Corollary 5]. □

The assumptions on the transformation S_ϵ guarantee that

$$\begin{aligned} D_\epsilon &\to [\nabla_y S_0]^{-1} D [\nabla_y S_0]^{-T} && \text{strongly in the two-scale sense,} \\ q_\epsilon &\to [\nabla_y S_0]^{-1} q_0 && \text{strongly in the two-scale sense,} \\ \epsilon^{-1} v_\epsilon &\to [\nabla_y S_0]^{-1} \partial_t S_0 && \text{strongly in the two-scale sense,} \end{aligned}$$

where the strong two-scale convergence is valid with respect to every L^p-norm for $p \in [1, \infty)$. To simplify the writing we define

$$D_0^* := [\nabla_y S_0]^{-1} D [\nabla_y S_0]^{-T}, \quad q_0^* := [\nabla_y S_0]^{-1} q_0, \quad v_0^* := [\nabla_y S_0]^{-1} \partial_t S_0$$

and state **Problem $\mathbf{P}_M$**, which is to find u_0 solving

$$\begin{aligned} \partial_t(J_0 u_0) - \nabla_y \cdot (J_0 D_0^* \nabla_y u_0 - J_0 q_0^* u_0 + J_0 v_0^* u_0) = J_0 f(u_0) \qquad & \text{in } (0,T) \times \Omega \times Y^*, \\ -(J_0 D_0^* \nabla_y u_0 - J_0 q_0^* u_0) \cdot \nu = -J_0 g(u_0) \qquad & \text{on } (0,T) \times \Omega \times \Gamma, \\ u_0(0) = u^0 \qquad & \text{in } \Omega \times Y^*, \\ u_0(t, x, \cdot) \text{ is } Y\text{-periodic.} & \end{aligned}$$

Due to the low regularity of $\partial_t u_0$, we cannot guarantee $u_0 \in C^0([0,T], L^2(\Omega \times Y^*))$, so the initial condition $u_0(0) = u^0$ holds only in a weaker sense. In fact, we show that there is a set of measure zero $N \subset (0,T)$, such that

$$\lim_{t \to 0,\, t \notin N} \|u_0(t) - u^0\|_{L^1(\Omega \times Y^*)} = 0.$$

Problem P_M is the macroscopic counterpart of Problem P_T, as follows from

Theorem 2. *The limit function $u_0 \in L^2((0,T)\times\Omega, H^1_{\text{per}}(Y^*))$ satisfies $\partial_t(J_0 u_0) \in L^2((0,T), L^2(\Omega, H^1_{\text{per}}(Y^*)'))$, and is the unique weak solution of Problem P_M.*

Proof. Testing in (4) with $\phi_\epsilon(t,x) := \phi\left(t, x, \frac{x}{\epsilon}\right)$ for $\phi \in C_0^\infty\left([0,T)\times\Omega, C^\infty_{\text{per}}(Y^*)\right)$ gives

$$\begin{aligned} &\int_0^T \langle \partial_t(J_\epsilon u_\epsilon), \phi_\epsilon \rangle_{\Omega_\epsilon} dt \\ &+ \int_0^T \int_{\Omega_\epsilon} \left[\epsilon^2 J_\epsilon D_\epsilon \nabla u_\epsilon - \epsilon J_\epsilon q_\epsilon u_\epsilon + J_\epsilon v_\epsilon u_\epsilon\right] \cdot \left[\nabla_x \phi\left(t, x, \frac{x}{\epsilon}\right) + \frac{1}{\epsilon} \nabla_y \phi\left(t, x, \frac{x}{\epsilon}\right)\right] dx dt \\ &= \int_0^T \int_{\Omega_\epsilon} J_\epsilon f(u_\epsilon) \phi_\epsilon dx dt + \epsilon \int_0^T \int_{\Gamma_\epsilon} J_\epsilon g(u_\epsilon) \phi_\epsilon d\sigma dt + \int_0^T \int_{\partial\Omega_\epsilon} J_\epsilon v_\epsilon \cdot \nu u_\epsilon \phi_\epsilon d\sigma dt. \end{aligned}$$

Using the convergence results from Proposition 5, we obtain for $\epsilon \to 0$

$$\begin{aligned} &\int_0^T \langle \partial_t(J_0 u_0), \phi \rangle_{L^2(\Omega, H^1_{\text{per}}(Y^*)'), L^2(\Omega, H^1_{\text{per}}(Y^*))} dt \\ &\quad + \int_0^T \int_\Omega \int_{Y^*} [J_0 D_0^* \nabla_y u_0 - J_0 q_0^* u_0 + J_0 \nabla v_0^* u_0] \cdot \nabla_y \phi dy dx dt \\ &= \int_0^T \int_\Omega \int_{Y^*} J_0 f(u_0) \phi dy dx dt + \int_0^T \int_\Omega \int_{Y^*} J_0 g(u_0) \phi d\sigma_y dx dt \\ &\quad + \int_0^T \int_\Omega \int_\Gamma J_0 v_0^* \cdot \nu u_0 \phi d\sigma_y dx dt.. \end{aligned}$$

The regularity of $J_0 u_0$ stated in Proposition 5 implies that $J_0 u_0 \in C^0([0,T], L^2(\Omega\times Y^*))$. Hence, integration by parts with respect to time in the equations above, and similar arguments as before, imply

$$(J_0 u_0)(0) = J_0(0)u^0 = u^0,$$

since $J_0(0) = 1$. Now, for almost every $t \in (0,T)$ we have

$$\|u_0(t) - u^0\|_{L^1(\Omega\times Y^*)} \leq \|u_0(t) - J_0(t)u_0(t)\|_{L^1(\Omega\times Y^*)} + \|J_0(t)u_0(t) - u^0\|_{L^1(\Omega\times Y^*)}.$$

The second term vanishes for $t \to 0$, due to the observations above. For the first term we obtain using $u_0 \in L^\infty((0,T), L^2(\Omega \times Y^*))$

$$\begin{aligned}\|u_0(t) - J_0(t)u_0(t)\|_{L^1(\Omega\times Y^*)} &\leq \|1 - J_0(t)\|_{L^1(\Omega\times Y^*)}\|u_0(t)\|_{L^2(\Omega\times Y^*)}\\ &\leq C\|1 - J_0(t)\|_{L^1(\Omega\times Y^*)} \xrightarrow{t\to 0} 0.\end{aligned}$$

□

5.1 Transformation to the evolving macroscopic domain

Now we formulate the macroscopic Problem P_M in an evolving macroscopic domain. We define $Y(t,x) := S_0(t,x,Y^*)$ and $\Gamma(t,x) := \partial Y(t,x) \setminus \partial Y$, and

$$\tilde{u}_0 : \bigcup_{(t,x)\in(0,T)\times\Omega} \{(t,x)\} \times Y(t,x) \to \mathbb{R}, \quad \tilde{u}_0(t,x,y) = u_0\left(t,x,S_0^{-1}(t,x,y)\right).$$

Further, we set for $(t,x) \in (0,T)\times\Omega$ and $y \in Y(t,x)$

$$\tilde{D}_0(t,x,y) := D_0^*\left(t,x,S_0^{-1}(t,x,y)\right), \quad \tilde{q}_0(t,x,y) := q_0^*\left(t,x,S_0^{-1}(t,x,y)\right),$$

and for $x \in \Omega$ and $y \in Y(0,x)$ the initial condition

$$\tilde{u}^0(x,y) := u^0\left(x, S_0^{-1}(0,x,y)\right).$$

Finally, let $Q^T := \bigcup_{(t,x)\in(0,T)\times\Omega}\{(t,x)\}\times Y(t,x)$ and $G^T := \bigcup_{(t,x)\in(0,T)\times\Omega}\{(t,x)\}\times\Gamma(t,x)$. With this, an elemental calculation shows that for all $\phi \in C^1\left(\overline{Q^T}\right)$ with $\phi(T,\cdot) = 0$ and Y-periodic, one has

$$\begin{aligned}&-\int_0^T\int_\Omega\int_{Y(t,x)} \tilde{u}_0\partial_t\phi dydxdt + \int_0^T\int_\Omega\int_{Y(t,x)}\left[\tilde{D}_0\nabla_y\tilde{u}_0 - \tilde{q}_0\tilde{u}_0\right]\cdot\nabla_y\phi dydxdt\\ &= \int_0^T\int_\Omega\int_{Y(t,x)} f(\tilde{u}_0)\phi dydxdt + \int_0^T\int_\Omega\int_{\Gamma(t,x)} g(\tilde{u}_0)\phi d\sigma_y dxdt\\ &\quad + \int_\Omega\int_{Y(0,x)}\tilde{u}^0\phi(0)dydx + \int_0^T\int_\Omega\int_{\Gamma(t,x)}\partial_t S_0\big(t, S_0(t,x,\cdot_y)^{-1}\big)\cdot\nu\tilde{u}_0\phi d\sigma_y dxdt.\end{aligned}$$

In other words, $\tilde{u}_0$ is the weak solution of the macroscopic problem defined in an evolving macroscopic domain

$$\begin{aligned}\partial_t\tilde{u}_0 - \nabla_y\cdot\left(\tilde{D}_0\nabla_y\tilde{u}_0 - \tilde{q}_0\tilde{u}_0\right) &= f(\tilde{u}_0) && \text{in } Q^T,\\ -\left(\tilde{D}_0\nabla_y\tilde{u}_0 - \tilde{q}_0\tilde{u}_0\right)\cdot\nu &= -g(\tilde{u}_0) && \text{on } G^T,\\ \tilde{u}_0(0) &= \tilde{u}^0 && \text{in } \bigcup_{x\in\Omega}\{x\}\times Y(0,x),\\ \tilde{u}_0 &\text{ is } Y\text{-periodic}.\end{aligned}$$

6 Conclusion

We derived a macroscopic model for a reaction-diffusion-advection problem defined in a domain with an evolving microstructure. The evolution is assumed known *a priori*. We consider a low diffusivity (of order ϵ^2), and include nonlinear bulk and surface reactions. The effective problem depends on the micro- and the macro-variable, and the evolution of the underlying microstructure is approximated by time- and space-dependent reference elements $Y(t,x)$. Hence, in each macroscopic point x, we have to solve a local cell problem on $Y(t,x)$. We emphasize that our methods are not restricted to the scalar case, but can be extended easily to systems of equations with Lipschitz-continuous nonlinearities. In order to carry out the homogenization limit, we proved general two-scale compactness results just based on a priori estimates for sequences of functions with oscillating gradients, and low regularity with respect to time. In doing so, we used the appropriately scaled function space $\mathcal{H}_\epsilon$, which allowed us to show compactness results, especially regarding the time-derivative.

In general applications, however, the evolution of the microstructure is not known *a priori*, and can be influenced by adsorption and desorption processes at the microscopic surface, or by mechanical forces. In such cases, one has to consider strongly coupled systems of transport, elasticity, and fluid flow equations in domains with an evolving microstructure, leading to highly nonlinear problems with free boundaries. In this context, the identification of the transformation S_ϵ and its control with respect to the parameter ϵ plays a crucial role. If this is achieved, the multi-scale methods developed in this paper can be employed in the study of such more complex applications.

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