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Abstract We consider a model for two-phase flow in a porous medium posed in a domain consisting of two adjacent regions. The model includes dynamic capillarity and hysteresis. At the interface between adjacent subdomains, the continuity of the normal fluxes and pressures is assumed. For finding the semi-discrete solutions after temporal discretization by the θ-scheme, we proposed an iterative scheme. It combines a (fixed-point) linearization scheme and a non-overlapping domain decomposition method. This article describes the scheme, its convergence and a numerical study confirming this result. The convergence of the iteration towards the solution of the semi-discrete equations is proved independently of the initial guesses and of the spatial discretization, and under some mild constraints on the time step. Hence, this scheme is robust and can be easily implemented for realistic applications.

1 Introduction

Flow in porous media has become a significant field of research, as prominent applications such as CO_2 storage and enhanced oil recovery vitally depend on the understanding of the underlying phenomena. Since measurements below surface are costly, if feasible at all, mathematical modeling and simulation are crucial to predict such processes. These models usually consist of coupled nonlinear differential equations, which may degenerate and change type. Besides the increasing complexity

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of the models incorporating dynamic capillarity and hysteresis, another difficulty is caused by the largely varying or even discontinuous physical properties.

To solve the coupled nonlinear equations, discretization and linearization schemes are necessary. Since Newton based solvers suffer from severe constraints on the time step sizes to ensure convergence [19], a simple fixed point iteration, the L-type linearization, has been proposed. Its high robustness comes at the price of a slower, linear convergence. Additionally, this approach is typically independent of the spatial discretization, and has thus been combined e.g. with (M)FEM [13, 18] or a discontinuous Galerkin method [10].

In the situation of block-heterogeneous soils, the application of a domain decomposition method seems natural to decouple the different homogeneous blocks and speed up the convergence. This approach is used and optimized for a wide range of applications [4, 8, 9, 7]. In [15], a non-overlapping Schwarz waveform-relaxation was analyzed for nonlinear convection-diffusion equations in a time-continuous setting. Such methods can also be used after temporal discretization for porous media equations [1, 5]. In [21, 22], the domain decomposition was integrated in the linearization process for the Richards equation respectively two-phase flow.

Here, we propose such a linearization and domain decomposition scheme for two-phase flow in porous media, including dynamic and hysteretic effects in the capillary pressure. These methods are independent of the chosen spatial discretization and avoid the use of derivatives as in Newton based iterations.

2 Mathematical Model and Temporal Discretization

Below, $T > 0$ is a fixed, final time and $\Omega \subset \mathbb{R}^d$ ($d \in \mathbb{N}$) a Lipschitz domain. It is partitioned into two Lipschitz subdomains Ω_1 and Ω_2 separated by a $(d-1)$-dimensional manifold Γ. The outer normal vectors at $\partial\Omega_l$ for $l \in \{1,2\}$ are denoted by $\boldsymbol{\nu}_l$. In each subdomain Ω_l, the flow of two immiscible, incompressible phases $\alpha \in \{n, w\}$ through a rigid porous medium is governed by the mass balance equations, the extended Darcy law and an extended, play-type capillary pressure model [2],

$$-\phi_l \partial_t s_l + \nabla \cdot \mathbf{u}_{n,l} = 0, \qquad \phi_l \partial_t s_l + \nabla \cdot \mathbf{u}_{w,l} = 0 \qquad \text{in } \Omega_l \times (0,T), \tag{1}$$

$$\mathbf{u}_{\alpha,l} = -\lambda_{\alpha,l}(s_l)\mathsf{K}_l \nabla p_{\alpha,l} \qquad \text{in } \Omega_l \times (0,T), \tag{2}$$

$$p_{n,l} - p_{w,l} = p_{c,l}(s_l) - \Phi_{\delta,l}(\partial_t s_l) - \partial_t T_l(s_l) \qquad \text{in } \Omega_l \times (0,T). \tag{3}$$

At Γ, the coupling conditions are the continuity of the normal fluxes and pressures

$$\mathbf{u}_{\alpha,1} \cdot \boldsymbol{\nu}_1 = -\mathbf{u}_{\alpha,2} \cdot \boldsymbol{\nu}_2, \qquad p_{\alpha,1} = p_{\alpha,2} \qquad \text{on } \Gamma \times (0,T). \tag{4}$$

Here, s_l denotes the saturation of the wetting phase, $u_{\alpha,l}$ the specific discharge of the α-phase and $p_{\alpha,l}$ its pressure. The parameters are the porosity $\phi_l \in (0,1)$, the intrinsic permeability $\mathsf{K}_l \in \mathbb{R}^{d\times d}$, which is symmetric, positive definite and bounded, the relative mobility $\lambda_{\alpha,l}$ and the capillary pressure p_c, while T_l and $\Phi_{\delta,l}$ model the dynamic respectively hysteretic effects. In contrast to equilibrium models, in which $T_l = \Phi_{\delta,l} = 0$, this model can reproduce experimental results such as fingering and saturation overshoots [17, 20]. Typically, (3) is a multi-valued relation $p_{n,l} - p_{w,l} \in p_{c,l}(s_l) - \gamma_l \,\mathrm{sign}(\partial_t s_l) - \partial_t T_l(s_l)$ involving a parameter $\gamma_l \geq 0$ and the sign graph. Here, we use a regularization $\Phi_{\delta,l}$ of sign; namely $\Phi_{\delta,l}(\xi) := \max\{-1, \min\{\delta^{-1}\xi, 1\}\}$ with $\delta > 0$ being a regularization parameter.

For simplicity, we only consider homogeneous Dirichlet boundary conditions for the pressures, i.e. $p_{w,l} \equiv p_{n,l} \equiv 0$ on $(\partial\Omega_l \cap \partial\Omega) \times (0, T)$. Together with an initial datum $s_l(0, \cdot) = s_l^0 \in L^\infty(\Omega)$, (1)–(4) form an initial-boundary-value problem in s, p_n and p_w.

Remark 1 For the existence of unique weak solutions to (1)–(3), we refer to [11, 6]. In particular, we mention [6] for the Hölder continuity of the pressure gradients $\nabla p_n, \nabla p_w$.

Notation 1 *We denote by $L^2(X)$, $H^1(X)$ and $H^{div}(X)$ the standard Hilbert spaces over $X \in \{\Omega, \Omega_1, \Omega_2\}$. $H^{1/2}(\Gamma)$ contains the traces $u|_\Gamma$ on Γ of functions $u \in H^1(\Omega)$. For the two subdomains Ω_l with $l \in \{1, 2\}$, the following spaces will be used*

$$\mathcal{W}_l := \{w \in H^1(\Omega_l) \,:\, w|_{\partial\Omega_l \cap \partial\Omega} \equiv 0\},$$
$$\mathcal{W} := L^2(\Omega) \times [\mathcal{W}_1 \times \mathcal{W}_2]^2, \quad \mathcal{V} := L^2(\Omega) \times [H_0^1(\Omega)]^2.$$

For any function $f \in L^2(\Omega)$, we denote by $f_l := f|_{\Omega_l}$ the restriction to Ω_l for $l \in \{1, 2\}$. Vice versa, we identify a pair of functions $(f_1, f_2) \in L^2(\Omega_1) \times L^2(\Omega_2)$ with f and consider f as the natural L^2-extension on the whole domain Ω. The L^2 inner product on Ω_1 or Ω_2 is denoted by $(\cdot, \cdot)$, while on Γ it is $(\cdot, \cdot)_\Gamma$.

Next, we summarize all assumptions on the coefficient functions, which are mostly also found in realistic physical systems. Note that the degeneration of the equations is excluded by requiring positive λ_α and Lipschitz continuous p_c. This can be enforced, if necessary, by a regularization like in [6, 16].

Assumption 1 For $l \in \{1, 2\}$ and $\alpha \in \{n, w\}$ we assume that

- $\lambda_{\alpha,l} : \mathbb{R} \to \mathbb{R}^+$ is Lipschitz continuous with Lipschitz constant $L_{\lambda_\alpha,l}$ and there exist $m_{\lambda_\alpha,l}, M_{\lambda_\alpha,l} \in \mathbb{R}^+$ such that $0 < m_{\lambda_\alpha,l} \leq \lambda_{\alpha,l}(s) < M_{\lambda_\alpha,l}$ for all $s \in \mathbb{R}$;
- $p_{c,l} : \mathbb{R} \to \mathbb{R}$ is strictly monotonically decreasing and there exist $m_{p_c,l}, L_{p_c,l} \in \mathbb{R}^+$ such that $m_{p_c,l}\,|r - s| \leq |p_{c,l}(r) - p_{c,l}(s)| \leq L_{p_c,l}\,|r - s|$ for all $r, s \in \mathbb{R}$;
- $T_l : \mathbb{R} \to \mathbb{R}$ is strictly monotonically increasing with Lipschitz constant $L_{T,l}$.

Remark 2 The extension of $\lambda_{\alpha,l}$, $p_{c,l}$ and T_l to any values $s \in \mathbb{R}$ can be constructed naturally. This is necessary since the the solutions to the non-degenerated model need not to satisfy a maximum principle [16].
Furthermore, $\Phi_{\delta,l} : \mathbb{R} \to \mathbb{R}$ is monotonically increasing and Lipschitz continuous with Lipschitz constant $L_{\Phi_\delta,l} = \gamma_l/\delta$.

We discretize the equations in time by the implicit θ-scheme. Given $N \in \mathbb{N}$, let $\Delta t := \frac{T}{N}$ and $\theta \in (0, 1]$. The superscript $(\cdot)^k$ denotes the approximations of the quantities at time $t^k = k\Delta t$, in particular we have $\mathbf{u}^k_{\alpha,l} := -\lambda_{\alpha,l}(s^k_l)\mathsf{K}_l\nabla p^k_{\alpha,l}$ and $p^k_{c,l} := p_{c,l}(s^k_l)$. Time averaged quantities are given by $(\cdot)^{k,\theta} := \theta(\cdot)^k + (1-\theta)(\cdot)^{k-1}$. After testing, partial integration and summation over $l = 1,2$ using the continuity of the normal flux across Γ, we obtain the time-discrete counterparts of (1)–(4).

Problem 1 (Semi-discrete weak formulation)

Given $(s^{k-1}, p^{k-1}_n, p^{k-1}_w) \in \mathcal{V}$, find $(s^k, p^k_n, p^k_w) \in \mathcal{V}$ such that for all $(\psi_p, \psi_n, \psi_w) \in \mathcal{V}$ there holds

$$-\sum_{l=1}^{2} \phi_l \left(\tfrac{s^k_l - s^{k-1}_l}{\Delta t},\ \psi_{n,l}\right) = \sum_{l=1}^{2} \left(\mathbf{u}^{k,\theta}_{n,l},\ \nabla\psi_{n,l}\right), \tag{5}$$

$$\sum_{l=1}^{2} \phi_l \left(\tfrac{s^k_l - s^{k-1}_l}{\Delta t},\ \psi_{w,l}\right) = \sum_{l=1}^{2} \left(\mathbf{u}^{k,\theta}_{w,l},\ \nabla\psi_{w,l}\right), \tag{6}$$

$$\sum_{l=1}^{2} \left(p^{k,\theta}_{n,l} - p^{k,\theta}_{w,l},\ \psi_{p,l}\right) = \sum_{l=1}^{2} \left(p^{k,\theta}_{c,l} - \Phi_{\delta,l}\left(\tfrac{s^k_l - s^{k-1}_l}{\Delta t}\right) - \tfrac{T_l(s^k_l) - T_l(s^{k-1}_l)}{\Delta t},\ \psi_{p,l}\right). \tag{7}$$

Remark 3 (Well-definedness)

If $(s^k, p^k_n, p^k_w) \in \mathcal{V}$ is a solution to Problem 1, we have $p_{\alpha,1}|_\Gamma = p_{\alpha,2}|_\Gamma$ by the definition of $\mathcal{V}$. Since $s^k_l, s^{k-1}_l \in L^2(\Omega_l)$, testing (5) and (6) with arbitrary $\psi_{\alpha,l} \in C^\infty_0(\Omega_l)$ implies $\mathbf{u}^{k,\theta}_{\alpha,l} \in H^{\mathrm{div}}(\Omega_l)$. Therefore, the normal trace lemma [3, Lemma III.1.1] yields $\mathbf{u}^{k,\theta}_{\alpha,l} \cdot \boldsymbol{\nu}_l \in H^{1/2}(\partial\Omega_l)'$ and integration by parts in (5) and (6) implies $\mathbf{u}^{k,\theta}_{\alpha,1} \cdot \boldsymbol{\nu}_1 = -\mathbf{u}^{k,\theta}_{\alpha,2} \cdot \boldsymbol{\nu}_2$ in $H^{1/2}_{00}(\Gamma)'$.

Proving the existence of solutions to this problem lies out of the scope of this paper, but may be done analogously to the time-continuous case mentioned in Remark 1. By this, the time-discrete pressure gradients should be bounded.

3 Linearization and Domain Decomposition

To account for the possible discontinuities at the interface Γ, we decouple the problems in the subdomains. Following [12], we combine the interface conditions $\mathbf{u}^{k,\theta}_{\alpha,1} \cdot \boldsymbol{\nu}_1 = -\mathbf{u}^{k,\theta}_{\alpha,2} \cdot \boldsymbol{\nu}_2$ and $p^k_{\alpha,1} = p^k_{\alpha,2}$ by a parameter $\mathcal{L}_\Gamma \in (0,\infty)$ to obtain

$$g_{\alpha,3-l} = -2\mathcal{L}_\Gamma p^k_{\alpha,l} - g_{\alpha,l}, \qquad \text{where} \qquad g_{\alpha,l} := \mathbf{u}^{k,\theta}_{\alpha,l} \cdot \nu_l - \mathcal{L}_\Gamma p^k_{\alpha,l} \qquad \text{on } \Gamma.$$

This Robin-type formulation is equivalent to the original conditions for any $\mathcal{L}_\Gamma \neq 0$, cf. [22, Remark 1 & 2]. In the next step, we introduce a linearized, iterative scheme, where $i \in \mathbb{N}$ is the iteration index. Given the previous solution $(s^{k.i-1}, p^{k,i-1}_n, p^{k,i-1}_w)$ and (g^{i-1}_n, g^{i-1}_w), we define the linearized fluxes and interface conditions as

$$\mathbf{u}^{k,i}_{\alpha,l} := -\theta\lambda_{\alpha,l}(s^{k,i-1}_l)\mathsf{K}_l\nabla p^{k,i}_{\alpha,l} + (1-\theta)\mathbf{u}^{k-1}_{\alpha,l}, \quad g^i_{\alpha,l} := -2\mathcal{L}_\Gamma p^{k,i-1}_{\alpha,3-l} - g^{i-1}_{\alpha,3-l}.$$

In this way, (5) and (6) become linear and decouple into

$$-\phi_l\left(\tfrac{s_l^{k,i}-s_l^{k-1}}{\Delta t},\,\psi_{n,l}\right)=\left(\mathbf{u}_{n,l}^{k,i},\,\nabla\psi_{n,l}\right)-\left(\mathcal{L}_\Gamma p_{n,l}^{k,i}+g_{n,l}^{i},\,\psi_{n,l}\right)_\Gamma, \tag{8}$$

$$\phi_l\left(\tfrac{s_l^{k,i}-s_l^{k-1}}{\Delta t},\,\psi_{w,l}\right)=\left(\mathbf{u}_{w,l}^{k,i},\,\nabla\psi_{w,l}\right)-\left(\mathcal{L}_\Gamma p_{w,l}^{k,i}+g_{w,l}^{i},\,\psi_{w,l}\right)_\Gamma, \tag{9}$$

$$g_{\alpha,l}^{i}=-2\mathcal{L}_\Gamma p_{\alpha,3-l}^{k,i-1}-g_{\alpha,3-l}^{i-1}\quad \text{in } L^2(\Gamma). \tag{10}$$

Finally, we also linearize (7) by adding stabilization terms, which vanish in the limit if the iteration converges. For the latter, we use the parameters $\mathcal{L}_{p,l},\mathcal{L}_{\Phi,l},\mathcal{L}_{T,l}>0$ to account for the nonlinearity of the functions $p_{c,l}$, $\Phi_{l,\delta}$ and T_l. They must satisfy some mild constraints to ensure the convergence of the scheme, as shown below. With this, the linearized and stabilized counterpart of (7) reads

$$\begin{aligned}\left(p_{n,l}^{k,\theta,i}-p_{w,l}^{k,\theta,i},\,\psi_{p,l}\right)&=\left(\theta p_{c,l}(s_l^{k,i-1})+(1-\theta)p_{c,l}^{k-1}-\Phi_{\delta,l}\left(\tfrac{s_l^{k,i-1}-s_l^{k-1}}{\Delta t}\right),\,\psi_{p,l}\right)\\&\quad-\left(\tfrac{T_l(s_l^{k,i-1})-T_l(s_l^{k-1})}{\Delta t}+\left(\mathcal{L}_{p,l}+\tfrac{\mathcal{L}_{T,l}+\mathcal{L}_{\Phi,l}}{\Delta t}\right)\left(s_l^{k,i}-s_l^{k,i-1}\right),\,\psi_{p,l}\right),\end{aligned} \tag{11}$$

where $p_{\alpha,l}^{k,\theta,i}:=\theta p_{\alpha,l}^{k,i}+(1-\theta)p_{\alpha,l}^{k-1}$. The iteration reduces to solving

Problem 2 (Weak formulation of the LDD-scheme)
Given $(s^{k-1},p_n^{k-1},p_w^{k-1})\in\mathcal{V}$, $(s^{k,i-1},p_n^{k,i-1},p_w^{k,i-1})\in\mathcal{W}$ and $(g_n^{i-1},g_w^{i-1})\in[L^2(\Gamma)]^4$, find $(s^{k,i},p_n^{k,i},p_w^{k,i})\in\mathcal{W}$ and $(g_n^i,g_w^i)\in[L^2(\Gamma)]^4$ such that (8)–(11) hold for $l\in\{1,2\}$ and all $(\psi_p,\psi_n,\psi_w)\in\mathcal{W}$.

3.1 Existence of Solutions and Convergence

Here, we summarize the theoretical results for the LDD iteration. This comprises the existence of unique solutions to Problem 2, and the convergence of the iterative sequence. The proofs are generalizations of the ones given in [14] and use ideas from [10, 12, 21, 22]. We omit the details here.

Lemma 1 (Existence)
Problem 2 has a unique solution.

Theorem 1 (Convergence)
Assume that a solution $(s^k,p_n^k,p_w^k)\in\mathcal{V}$ of Problem 1 exists and satisfies $\|\mathsf{K}_l^{1/2}\nabla p_{\alpha,l}^k\|_{L^\infty(\Omega_l)}\le M_{p_\alpha,l}$ as well as $\mathbf{u}_{\alpha,l}^k\cdot\boldsymbol{\nu}_l\in L^2(\Gamma)$. Let Assumption 1 be fulfilled. If the stabilization parameters and time step fulfill for $l\in\{1,2\}$

$$\mathcal{L}_{p,l} \geq \frac{L_{p_c,l}}{\theta}, \quad \mathcal{L}_{T,l} \geq \frac{L_{T,l}}{2}, \quad \mathcal{L}_{\Phi,l} \geq \frac{L_{\Phi_\delta,l}}{2} \quad \textit{and} \quad \Delta t < \frac{\phi_l m_{p_c,l}}{\sum\limits_{\alpha\in\{n,w\}} \frac{\theta L^2_{\lambda_\alpha,l} M^2_{p_\alpha,l}}{m_{\lambda_\alpha,l}}},$$

the sequence of solutions of Problem 2 converges towards (s^k, p_n^k, p_w^k) *for any initial guess* $(s^{k,0}, p_n^{k,0}, p_w^{k,0}) \in \mathcal{W}$ *and* $(g_n^0, g_w^0) \in [L^2(\Gamma)]^4$*, i.e. for* $l \in \{1,2\}$ *and* $\alpha \in \{n,w\}$

$$s_l^{k,i} \to s_l^k \ \text{ in } \ L^2(\Omega_l), \quad p_{\alpha,l}^{k,i} \to p_{\alpha,l}^k \ \text{ in } \ \mathcal{W}_l, \quad g_{\alpha,l}^i \rightharpoonup g_{\alpha,l} \ \text{ in } \ L^2(\Gamma) \qquad \text{as } i \to \infty.$$

Remark 4 We have $L_{\Phi_\delta,l} = \gamma_l/\delta$, such that $\mathcal{L}_{\Phi,l} \geq \gamma_l/(2\delta)$, while the other parameters and the time step are independent of the regularization.

4 Numerical Experiment

For the validation of the theoretical results, we present a numerical study in a rectangular domain $\Omega = (-1, 1) \times (0,1)$ split into subdomains at the interface $\Gamma = \{0\} \times (0,1)$. We use a standard finite element method (Q_2) with a uniform mesh with mesh size Δx matching at the interface Γ. We choose the final time $T = 1$ and the Crank-Nicolson method ($\theta = 1/2$) in time, so that we expect errors of the order $O(\Delta t^2 + \Delta x^2)$. Furthermore, we take the same linearization parameters on both subdomains, i.e. $\mathcal{L}_f := \mathcal{L}_{f,1} = \mathcal{L}_{f,2}$ for $f \in \{p, T, \Phi\}$.

We consider an analytically solvable example with isotropic and constant absolute permeability $K_1 = K_2 = I$, and constant porosity $\phi_1 = \phi_2 = 1$ to explicitly compute the experimental order of convergence (EOC). We choose linear coefficient functions, but no hysteresis, i.e. $\lambda_n(s) = 1 - s$, $\lambda_w(s) = s$, $p_c(s) = 0.2 - s$, $T(s) = s$, and $\gamma \equiv 0$. The boundary conditions and right-hand side are selected such that the solution is

$$p_n(x,t) = \frac{(1-x_1)(1+x_1)^2}{2(1+t)^2}, \quad p_w(x,t) = \frac{(1-x_1)(1+x_1)^2}{2(1+t)}, \quad s(x,t) = \frac{(1-x_1)(1+x_1)^2}{2(1+t)} + 0.2.$$

First, we study the behavior of the method with respect to the time step and mesh size. The results in table 1 clearly confirm the second order convergence in Δt and Δx and indicate that the LDD-iteration is discretization independent, since the average number of iterations per time step stays almost constant.

Next, we study the convergence properties of the method within one time step. For fixed discretization, we study the error reduction and convergence rate in the last time step. The results in Fig. 1 indicate a fast, linear convergence. Moreover, a proper choice of the LDD parameters is crucial for the fast convergence, which can be seen in Fig. 2. Finding the optimum is an open problem, but the lower bounds from our analysis ($\mathcal{L}_p \geq 1/2$ and $\mathcal{L}_T \geq 1/2$) are reasonable indicators.

Table 1 The LDD-scheme with the parameters $\mathcal{L}_p = 0.5$, $\mathcal{L}_T = 1$ and $\mathcal{L}_\Gamma = 0.375$ ($\mathcal{L}_\Phi = 0$) achieves experimentally second order convergence (EOC) in pressure (p) and saturation (s). The average number of iterations per time step stays almost constant.

$\Delta t = \Delta x$	$\lVert e_p \rVert_{L^2H^1}$	EOC_p	$\lVert e_s \rVert_{L^2H^1}$	EOC_s	Avg. #Iter.
0.2	$5.352 \cdot 10^{-3}$		$5.824 \cdot 10^{-3}$		13
0.1	$1.394 \cdot 10^{-3}$	1.94	$1.463 \cdot 10^{-3}$	1.993	12.3
0.05	$3.564 \cdot 10^{-4}$	1.968	$3.670 \cdot 10^{-4}$	1.995	12
0.025	$9.013 \cdot 10^{-5}$	1.983	$9.192 \cdot 10^{-5}$	1.997	11.5
0.0125	$2.273 \cdot 10^{-5}$	1.987	$2.312 \cdot 10^{-5}$	1.991	15.5

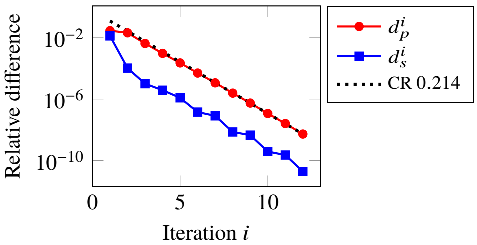

Fig. 1 Error reduction within the last time step of the LDD-scheme for $\Delta t = 0.05$ and $\Delta x = 0.05$. The relative L^2-differences d_p^i and d_s^i in pressure and saturation decrease fast, and the fitted convergence rate (CR) is low.

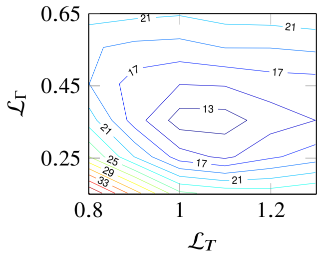

Fig. 2 Parameter dependence of the average number of iterations per time step for fixed $\Delta t = \Delta x = 0.05$ (For simplicity $\mathcal{L}_p = 0$). Deviations from the optimal parameter set drastically increase the convergence rate.

5 Conclusion

We proposed an iterative LDD-scheme for finding the semi-discrete solutions of a non-equilibrium two-phase model in a block-heterogeneous domain. We summarized the existence and convergence of the solutions of this LDD-scheme, which holds under a mild restriction for the time step, independently of the initial guesses or of the used spatial discretization. Therefore, the scheme is robust and can be easily adapted for realistic applications.

We will provide a detailed analysis and further numerical studies in a follow-up article. Further investigation is necessary to generalize the method for the degenerated cases. Moreover, an a-posteriori error analysis might lead to estimates for efficient and adaptive stopping criteria.

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